Solution

MATHS

JEE main - Mathematics

MATHS (Section-A)

1.

(c) R - {2}

Explanation:

$f(x)=\frac{2x^2-5x-3}{x^2-4x+3}$

$=\frac{(2x+1)(x-3)}{(x-1)(x-3)}$

$=\frac{2x+1}{x-1}$ = y (say)

$\Rightarrow$ 2x + 1 = xy - y $\Rightarrow$ 1 + y = (y - 2)x

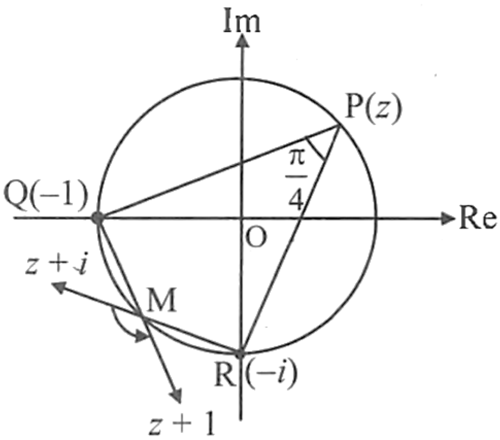
$\Rightarrow x=\frac{1+y}{y-2}$ = $f^{-1}(y)$, provided y $\neq$ 2

Range of f(x) = R - {2}

2.

(b) $\frac{3\pi}{4}$

Explanation:



|z| = 1 $\Rightarrow$ P(z) moves on the circle with centre at the origin and radius = 1

arg $\left(\frac{z+1}{z+i}\right)$ = angle by which vector z + i is rotated to get into the direction of z + 1

$\angle$QPR = $\frac{\pi}{4}$ $\Rightarrow$ $\angle$QMR = $\frac{3\pi}{4}$

$\Rightarrow$ arg$\left(\frac{z+1}{z+i}\right)=\frac{3\pi}{4}$

3. **(a)** 151200

Explanation:

There are 6 consonants (IN, IR, ID, IM, and 2T's) and 6 vowels (2I's, 3E's, and 1A)

Number of ways to arrange 6 consonants = $\frac{6!}{2!}$

Now, there are 7 gaps (available to arrange 6 vowels) created by these 6 consonants

Number of ways to arrange 6 vowels in these 7 gaps = $\frac{^7P_6}{2!3!}$

$\Rightarrow$ The required number of arrangements

$=\frac{6!}{2!}\times\frac{^7P_6}{2!3!}$ = 360 $\times$ 420 = 151200

4. **(a)** $\frac{1}{16}$(1 + 7x)

Explanation:

$\frac{\left(1+\frac{2}{3}x\right)^{-3}(1-15x)^{\frac{-1}{5}}}{(2-3x)^4} = \frac{\left(1+\frac{2}{3}x\right)^{-3}(1-15x)^{\frac{-1}{5}}}{2^4\left(1-\frac{3x}{2}\right)^4} = \frac{(1-2x)(1+3x)}{16(1-6x)}$, ...[neglecting x_2 and higher powers of x]

$= \frac{(1+x-6x^2)(1-6x)^{-1}}{16}$

= $\frac{1}{16}$ (1 + x - 6x^2) (1 + 6x + ...)

= $\frac{1}{16}$(1 + 7x)

5. **(a)** 1

Explanation:

Let a and b be two numbers.

Sum of n A.M.'s = n $\times$ A.M. of a and b

$\Rightarrow$ A_1 + A_2 = 2 $\times$ $\left(\frac{a+b}{2}\right)$ = a + b

Product of n G.M.'s = (G.M. of a and b)^n

$\Rightarrow G_1 \cdot G_2 = (\sqrt{ab})^2$ = ab

$\Rightarrow \frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{b}$ are in A.P.

$\Rightarrow \frac{1}{H_1} + \frac{1}{H_2} = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$

$\Rightarrow \frac{H_1H_2}{H_1+H_2} = \frac{G_1G_2}{A_1+A_2}$

$\Rightarrow \frac{G_1G_2}{H_1H_2} \times \frac{H_1+H_2}{A_1+A_2}$ = 1

6. **(a)** -8

Explanation:

Given function $f(x) = \begin{cases} x+a, & x \le 0 \\ |x-4|, & x > 0 \end{cases}$

$g(x) = \begin{cases} x+1, & x < 0 \\ (x-4)^2 + b, & x \ge 0 \end{cases}$

Function is continuous at R, and it is defined at x = 0.

Then, L.H.L. = R.H.L. = f(x).

L.H.L.

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0} f(x)$

$\lim\limits_{x\to 0^+} |x-4| = \lim\limits_{x\to 0}(x+a)$

$\lim\limits_{h\to 0}|(0+h)-4| = (0+a)$

a = 4.

R.H.L.

$\lim\limits_{x\to 0^+} g(x) = \lim\limits_{x\to 0^-} g(x)$

$\lim\limits_{x\to 0^+} \left[(x-4)^2 + b\right]$

$= \lim\limits_{x\to 0^-} (x+1)$

16 + 6 = 1 $\Rightarrow$ b = - 15

Take, (gof)(2) + (fog)(-2)

g(f(2)) + f(g(-2)) = g(12 - 41) + f(-2 +1)

= g(2) + f(-1) = (2 - 4)^2 - 15 + (-1) + 4

= 4 - 15 - 1 + 4 = 8 - 16 = -8.

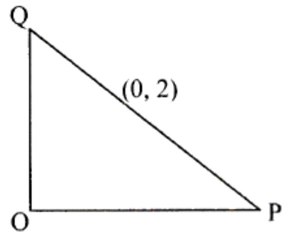
7.

(b) n(2e)$^{1/n}$

Explanation:

Given a_1a_2 ... a_n = e

Consider a_1a_2 ... a_{n-1} + a + 2a_n



AM $\geq$ GM

$\frac{a_1+a_2+\ldots+2a_n}{n} \geq (a_1a_2a_3 \ldots 2a_n)^{1/n}$

$= (2e)^{1/n}$

$\therefore a_1 + a_2 + \ldots + a_{n-1} + 2a_n \geq n\,(2e)^{1/n}$

8. **(a)** $\int_{a-1}^{b-1} f(x+1)dx$

Explanation:

$I = \frac{1}{(a+b)}\int_a^b x[f(x) + f(x+1)]dx$

$x \to a + b - x$ ...(i)

$I = \frac{1}{(a+b)}\int_a^b (a+b-x)[f(a+b-x) + f(a+b+1-x)]dx$

$I = \frac{1}{(a+b)}\int_a^b (a+b-x)\,[(x+1) + f(x)]\,dx$ ...(ii)

Add (i) and (ii) [$\because$ put $x \to x + 1$ in $f(a + b + 1 - x) = f(x)$]

$2I = \int_a^b [f(x+1) + f(x)]\,dx$

$2I = \int_a^b f(x+1)dx + \int_a^b f(x)\,dx$

$= \int_a^b f(a+b+1-x)dx + \int_a^b f(x)dx$

$2I = 2\int_a^b f(x)dx$

$I = \int_{a-1}^{b-1} f(x+1)dx$ [$\because$ Put $x \to x + 1$]

9.

(b) $\frac{2}{3}$

Explanation:

$\frac{2}{3}$

10.

(d) $2\sqrt{2}$

Explanation:

We know that,

It's a circle with center O,

So, we need a point to find the radius.

$x = \sqrt{7}$, the intersection $(1, \sqrt{7})$ is on the circle

The radius is

$= \sqrt{1^2 + \sqrt{7}^2}$

$= \sqrt{8}$

$= 2\sqrt{2}$

11.

(b) 6

Explanation:

6

12.

(d) 15

Explanation:

Let y = vx ⇒ $\frac{dy}{dx}$ = v + x$\frac{dv}{dx}$

now, given differentiable equation is

$\frac{dy}{dx}$ = $\frac{y}{x}+\frac{1}{x}\sqrt{y^2+16x^2}$ ⇒ x$\frac{dv}{dx}$ = $\sqrt{v^2+16}$

⇒ $\int\frac{dv}{\sqrt{v^2+16}}$ = $\int\frac{dx}{x}$ ⇒ $\ln\left|v+\sqrt{v^2+16}\right|$ = lnx + lnC

$\left|v+\sqrt{v^2+16}\right|$ = CX $\frac{y}{x}+\frac{\sqrt{y^2+16}}{x}$ = Cx

⇒ y + $\sqrt{y^2+16x^2}$ = Cx^2; As y(1) = 3 ⇒ C = 8

⇒ y(2) = 15

13.

(b) -4

Explanation:

Given, direction ratios are proportional to 2, - 1, 2

⇒ Direction cosines = $(2\lambda,-\lambda,2\lambda)$

l^2 + m^2 + n^2 = 1

⇔ $(2\lambda)^2+(-\lambda)^2+(2\lambda)^2$ = 1

⇔ $9\lambda^2$ = 1 ⇔ λ = $\pm\frac{1}{3}$

⇒ Direction cosines are $\left(\pm\frac{2}{3},\mp\frac{1}{3},\pm\frac{2}{3}\right)$

But it makes obtuse angle with x-axis.

⇒ l = $-\frac{2}{3}$

⇒ The component of $\vec{r}$ along x-axis

= 6 × $\left(-\frac{2}{3}\right)$ = -4

14.

(d) I, II and III

Explanation:

$\frac{|\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}|}{\overrightarrow{BA}\cdot\overrightarrow{BC}}+\frac{|\vec{c}\times\vec{d}+\vec{d}\times\vec{a}+\vec{a}\times\vec{c}|}{\overrightarrow{DA}\cdot\overrightarrow{DC}}=0$

⇔ $\frac{2\text{ area }\Delta ABC}{|\overrightarrow{BA}||\overrightarrow{BC}|\cos B}+\frac{2\text{ area }\Delta ADC}{|\overrightarrow{DA}||\overrightarrow{DC}|\cos D}=0$

⇔ $\frac{2\text{ area }\Delta ABC}{|\overrightarrow{BA}||\overrightarrow{BC}|}\cdot\frac{\tan B}{\sin B}+\frac{2\text{ area }\Delta ADC}{|\overrightarrow{DA}||\overrightarrow{DC}|}\cdot\frac{\tan D}{\sin D}=0$

⇔ $\frac{2\text{ area }\Delta ABC}{2\text{ area }\Delta ABC}\cdot\tan B+\frac{2\text{ area }\Delta ADC}{2\text{ area }\Delta ADC}\cdot\tan D=0$

⇔ tan B + tan D = 0

⇔ sin(B + D) = 0 ⇔ B + D = π

⇔ ☐ABCD is cyclic

15.

(c) 833

Explanation:

First 50 even natural numbers are 2, 4, 6, ..., 100

Variance = $\frac{\sum x_i^2}{N} - (\bar{x})^2$

$\Rightarrow \sigma^2 = \frac{2^2+4^2+\ldots+100^2}{50} - \left(\frac{2+4+\ldots+100}{50}\right)^2$

= $\frac{4(1^2+2^2+3^2+\ldots..+50^2)}{50}$ - $(51)^2$

= $4\left(\frac{50\times51\times101}{50\times6}\right)$ - $(51)^2$

= 3434 - 2601 $\Rightarrow \sigma^2$ = 833

16.

(b) $\frac{7}{16}$

Explanation:

We have, P (exactly one of A or B occurs)

= P(A ∪ B) - P(A ∩ B)

= P(A) + P(B) -2P(A ∩ B)

According to the question,

$P(A) + P(B) - 2P(A \cap B) = \frac{1}{4}$ ...(i)

$P(B) + P(C) - 2P(B \cap C) = \frac{1}{4}$ ...(ii)

and $P(C) + P(A) - 2P(C \cap A) = \frac{1}{4}$ ...(iii)

On adding Eqs. (i), (ii) and (iii), we get

2 [P(A) + P(B) + P(C) - P(A ∩ B) - P(B ∩ C) - P(C ∩ A)] = $\frac{3}{4}$

$\Rightarrow$ P(A) + P(B) + P(C) - P(A ∩ B) - P(B ∩ C) - P(C ∩ A) = $\frac{3}{8}$

$\therefore$ P (atleast one event occurs)

= P (A ∪ B ∪ C)

= P(A) + P(B) + P(C) - P(A ∩ B) - P(B ∩ C) - P(C ∩ A) + P(A ∩ B ∩ C) $\left[\because P(A \cap B \cap C) = \frac{1}{16}\right]$

$= \frac{3}{8} + \frac{1}{16} = \frac{7}{16}$

17.

(c) $\frac{7}{5}$

Explanation:

Let $\cos\alpha + \cos\beta = \frac{3}{2}$

$\Rightarrow 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = \frac{3}{2}$ ...(i)

and $\sin\alpha + \sin\beta = \frac{1}{2}$

$\Rightarrow 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = \frac{1}{2}$ ...(ii)

On dividing (ii) by (i), we get

$\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{1}{3}$

Given: $\theta = \frac{\alpha+\beta}{2} \Rightarrow 2\theta = \alpha + \beta$

Consider $\sin 2\theta + \cos 2\theta = \sin(\alpha+\beta) + \cos(\alpha+\beta)$

$= \frac{\frac{2}{3}}{1+\frac{1}{9}} + \frac{1-\frac{1}{9}}{1+\frac{1}{9}} = \frac{6}{10} + \frac{8}{10} = \frac{7}{5}$

18. **(a)** $\sqrt{2}$

Explanation:

Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2}$ = 1,

Point F is the focus on the positive X-axis and AB is the latus rectum.

The coordinates of F are (ae, 0) and the coordinates of the ends of the latus rectum, AF are BF are (ae, $\frac{\mathbf{b}^2}{\mathbf{a}}$) and (ae, $-\frac{\mathbf{b}^2}{\mathbf{a}}$) respectively.

It is given that the latus rectum subtends an angle of 90^o at the centre.

$\angle$AOF = 45^o

since $\triangle$AOB $\cong$ $\angle$BOF

$\angle$OAF = 45° since $\angle$OFA = 90°
OF = AF
But OF = ae
AF = $\frac{b^2}{a}$ = ae
e = $\frac{b^2}{a^2}$
e = $\sqrt{2}$

19.
(b) 55
Explanation:
Let n(U) = 100, then n(A) = 63, n(B) = 76
n(A ∩ B) = x
Now, n($A \cup B$) = n(A) + n(B) - n(A ∩ B) $\leq$ 100
= 63 + 76 - x $\leq$ 100
$\Rightarrow$ x $\geq$ 139 - 100 $\Rightarrow$ x $\geq$ 39
$\because$ n(A ∩ B) $\leq$ n(A) $\Rightarrow$ x $\leq$ 63
$\therefore 39 \leq x \leq 63$

20.
(c) $\frac{1}{16}$
Explanation:
Given, $|ABA^T| = 8$
$\Rightarrow |A|\,|B|\,|A^T| = 8$ [$\because |XY| = |X|\,|Y|$]
$\therefore |A|^2\,|B| = 8$ ...(i) [$\because |A^T| = |A|$]
Also, we have $|AB^{-1}| = 8 \Rightarrow |A|\,|B^{-1}| = 8$
$\Rightarrow \frac{|A|}{|B|} = 8$ ...(ii) $\left[\because \left|A^{-1}\right| = |A|^{-1} = \frac{1}{|A|}\right]$
On multiplying Eqs. (i) and (ii), we get
$|A|^3 = 8 \cdot 8 = 4^3$
$\Rightarrow |A| = 4$
$\Rightarrow \quad |B| = \frac{|A|}{8} = \frac{4}{8} = \frac{1}{2}$
Now, $\left|BA^{-1}B^T\right| = |B|\frac{1}{|A|}|B| = \left(\frac{1}{2}\right)\frac{1}{4}\left(\frac{1}{2}\right) = \frac{1}{16}$

MATHS (Section-B)

21. 2
Explanation:
Let $f_n(x) = \int\limits_x^{2x} e^{-t^n}\,dt$
$f_n'(x) = 2 \cdot e^{-(2x)^n} - e^{-x^n}$
For maxima and minima, $f_n'(x) = 0 \Rightarrow 2e^{-(2x)^n} = e^{-x^n} \Rightarrow 2 \cdot e^{-(2^n x^n)} = e^{-x^n}$
Taking log on both sides, we get,
$\ln 2 - 2^n x^n = -x^n \Rightarrow \ln 2 = x^n (2^n - 1) \Rightarrow x^n = \frac{\ln 2}{2^n - 1} \Rightarrow x = \left(\frac{\ln 2}{2^n-1}\right)^{\frac{1}{n}} = a_n$
Also, $f_n''\left(x = \left(\frac{\ln 2}{2^n-1}\right)^{\frac{1}{n}}\right) < 0 \Rightarrow f_n(x)$ is maximum at $x = \left(\frac{\ln 2}{2^n-1}\right)^{\frac{1}{n}}$
Now, $\ln a_n = \frac{\ln\left(\frac{\ln 2}{2^n-1}\right)}{n} = \frac{\ln(\ln 2) - \ln(2^n-1)}{n}$
Hence $L = \lim\limits_{n\to\infty} \ln(a_n) = \lim\limits_{n\to\infty} \frac{\ln(\ln 2) - \ln(2^n-1)}{n} = \lim\limits_{n\to\infty}\left(\frac{\ln(\ln 2)}{n} - \frac{\ln\left(2^n\left(1 - \frac{1}{2^n}\right)\right)}{n}\right)$
$L = \lim_{n\to\infty}\left(0 - \frac{n \cdot \ln 2 + \ln\left(1 - \frac{1}{2^n}\right)}{n}\right) = -\ln 2$
Hence, $e^{-L} = 2$

22. 3.0

Explanation:

$\overrightarrow{p} + \overrightarrow{q} = 3\hat{i} + 5\hat{j} + 2\hat{k}$ and $\overrightarrow{p} - \overrightarrow{q} = \hat{i} + \hat{j}$

Now, $(\overrightarrow{p} + \overrightarrow{q}) \times (\overrightarrow{p} - \overrightarrow{q}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 5 & 2 \\ 1 & 1 & 0 \end{vmatrix} = -2\hat{i} + 2\hat{j} - 2\hat{k}$

$\therefore \vec{r} = \lambda(-2\hat{i} + 2\hat{j} - 2\hat{k})$

$|\vec{r}| = \pm 2\sqrt{3\lambda} = \sqrt{3} \Rightarrow \lambda = \pm\frac{1}{2}$

$\therefore |\alpha| + |\beta| + |\gamma|$ = 1 + 1 + 1 = 3

23. 4

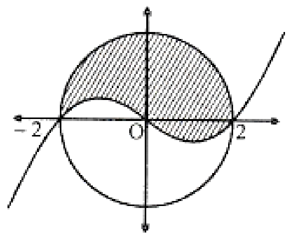
Explanation:

f'(x) = a$\left(x^2 - \frac{4}{3}\right) \Rightarrow$ f(x) = a$\left(\frac{x^3}{3} - \frac{4x}{3}\right)$ + b passes through (0, 0) and (1, -2).

$\therefore$ b = 0, a = 2

f(x) = $\frac{2x}{3}$(x^2 - 4)

Required area = $\frac{\pi(2)^2}{2} = 2\pi = 4\frac{\pi}{2} \Rightarrow$ k = 4



24. 10.0

Explanation:

Since, given f(1) = $\frac{1}{5}$

$\therefore f(2) = f(1) + f(1) = \frac{2}{5} \Rightarrow f(2) = \frac{2}{5}$

$f(3) = f(2) + f(1) = \frac{3}{5} \Rightarrow f(3) = \frac{3}{5}$

As so on we get $f(n) = \frac{n}{5}$

$\therefore \sum_{n=1}^{m} \frac{f(n)}{n(n+1)(n+2)} = \frac{1}{5}\sum_{n=1}^{m}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$

$= \frac{1}{5}\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{m+1} - \frac{1}{m+2}\right)$

$= \frac{1}{5}\left(\frac{1}{2} - \frac{1}{m+2}\right) = \frac{m}{10(m+2)} = \frac{1}{12}$

$\therefore$ m = 10

25. 100.0

Explanation:

Given X = $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

thus X^2 = $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Y = $\begin{bmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{bmatrix}$, Z = $\begin{bmatrix} \alpha^2 & -\alpha\beta & \beta^2 - \alpha\gamma \\ 0 & \alpha^2 & -\alpha\beta \\ 0 & 0 & \alpha^2 \end{bmatrix}$

Y$\cdot$Y^{-1} = I

$\begin{bmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{bmatrix} \begin{bmatrix} \frac{1}{5} & \frac{-2}{5} & \frac{1}{5} \\ 0 & \frac{1}{5} & \frac{-2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\frac{\alpha}{5}$ = 1 $\Rightarrow \alpha$ = 5

$-\frac{2}{5}\alpha + \frac{\beta}{5}$ = 0 $\Rightarrow \beta$ = 10

$\frac{\alpha}{5} - \frac{2\beta}{5} + \frac{\gamma}{5} = 0 \Rightarrow \gamma = 15$

$\Rightarrow (\alpha - \beta + \gamma)^2 = (5 - 10 + 15)^2 = 100$