Solution

MATHS

JEE main - Mathematics

MATHS (Section-A)

1. **(a)** $f^{-1}(x) = f(x)$

Explanation:

Let y = f(x) $= \frac{5x+3}{7x-5}$

$\Rightarrow$ 5x + 3 = 7xy - 5y

$\Rightarrow x = \frac{5y+3}{7y-5}$

$\Rightarrow f^{-1}(y) = \frac{5y+3}{7y-5}$

$\Rightarrow f^{-1}(x) = \frac{5x+3}{7x-5}$ = f(x)

2.

(b) the imaginary axis

Explanation:

$\frac{3+2z-z^2}{z-1} = \frac{4-(z-1)^2}{z-1} = \frac{|z-1|^2-(z-1)^2}{z-1}$

$= \frac{(z-1)(\overline{z-1})-(z-1)^2}{z-1}$

$= \overline{(z-1)}$ - (z - 1)

= a purely imaginary number

$\Rightarrow$ All the values of $\frac{3+2z-z^2}{z-1}$ lie on the imaginary axis.

3. **(a)** 6

Explanation:

The last two digits must form a number divisible by 4 viz. 12, 24, 32

can be filled in 3 ways

1^{st} place can be filled in by one of the remaining two digits and 2nd place can be filled in one way only.

Required number of four-digit numbers

= 2 $\times$ 1 $\times$ 3 = 6

4.

(d) (28, 315)

Explanation:

Given expression is $(1 + ax + bx^2)(1 - 3x)^{15}$. In the expansion of binomial $(1 - 3x)^{15}$, the (r + 1)th term is

$T_{r+1} = {}^{15}C_r(-3x)^r = {}^{15}C_r(-3)^r x^r$

Now, coefficient of x^2, in the expansion of $(1 + ax + bx^2)(1 - 3x)^{15}$ is

${}^{15}C_2(-3)^2 + a\,{}^{15}C_1(-3)^1 + b\,{}^{15}C_0(-3)^0 = 0$ (given)

$\Rightarrow$ (105 $\times$ 9) - 45a + b = 0

$\Rightarrow$ 45a - b = 945 ...(i)

Similarly, the coefficient of x^3, in the expansion of

$(1 + ax + bx^2)(1 - 3x)^{15}$ is

${}^{15}C_3(-3)^3 + a\,{}^{15}C_2(-3)^2 + b\,{}^{15}C_1(-3)^1 = 0$ (given)

$\Rightarrow$ -12285 + 945a - 45b = 0

$\Rightarrow$ 63a - 3b = 819

$\Rightarrow$ 21a - b = 273 ...(ii)

From Eqs. (i) and (ii), we get

24a = 672 $\Rightarrow$ a = 28

So, b = 315

$\Rightarrow$ (a, b) = (28, 315)

5.

(d) A.P.

Explanation:

$\frac{1}{\sqrt{a}+\sqrt{b}} = \frac{\sqrt{b}-\sqrt{a}}{b-a}, \frac{1}{\sqrt{a}+\sqrt{c}} = \frac{\sqrt{c}-\sqrt{a}}{c-a}$ and $\frac{1}{\sqrt{b}+\sqrt{c}} = \frac{\sqrt{c-\sqrt{b}}}{c-b}$

a, b, c are in A.P. If d is the common ratio, then b - a = d, c - a = 2d and c - b = d

The given three numbers are $\frac{\sqrt{b}-\sqrt{a}}{d}, \frac{\sqrt{c}-\sqrt{a}}{2d}$ and $\frac{\sqrt{c}-\sqrt{b}}{d}$

$\Rightarrow \frac{\sqrt{b}-\sqrt{a}}{d} + \frac{\sqrt{c}-\sqrt{b}}{d} = \frac{\sqrt{c}-\sqrt{a}}{d}$

$\therefore$ The three numbers are in A.P.

6.

(c) 5

Explanation:

||x| - 1| = |-x - 1|, x < 0

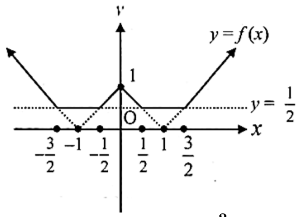
= |x - 1|, x $\geq$ 0

||x| - 1| = -x - 1, x < -1

= x + 1, -1 < x < 0

= -x + 1, 0 $\leq$ x < 1

= x - 1, x $\geq$ 1



f(x) = -x - 1, $x < -\frac{3}{2}$

$= \frac{1}{2}, -\frac{3}{2} \leq x \leq -\frac{1}{2}$

= x + 1, $-\frac{1}{2} \leq$ x < 0

= -x + 1, $0 \leq x < \frac{1}{2}$

$= \frac{1}{2}, \frac{1}{2} \leq x < \frac{3}{2}$

= x - 1, $x \geq \frac{3}{2}$

The point at which f may not be differentiable are $x = -\frac{3}{2}, -\frac{1}{2}, 0, \frac{1}{2}, \frac{3}{2}$.

Differentiability of function f at $x = -\frac{3}{2}$

L.h. lim $= \lim\limits_{x \to -\frac{3}{2}} \frac{(-x-1)-\frac{1}{2}}{x+\frac{3}{2}} = -1$

R. h. lim $= \lim\limits_{x \to -\frac{3^+}{2}} \frac{\frac{1}{2}-\frac{1}{2}}{x+\frac{3}{2}} = 0$

L. h. lim $\neq$ R. h. lim

$\Rightarrow$ f is not differentiable at $x = -\frac{3}{2}$

	$x = -\frac{1}{2}$	x = 0	$x = \frac{1}{2}$	$x = \frac{3}{2}$
L. h. lim	0	1	-1	0
R. h. lim	1	-1	0	1

From the table given above, f is not differentiable at $x = -\frac{1}{2}, 0, \frac{1}{2}, \frac{3}{2}$.

7. **(a)** $\frac{1}{18\pi}$

Explanation:

Let the thickness of layer of ice is x cm, the volume of spherical ball (only ice layer) is

V = $\frac{4}{3}\pi$ [(10 + x)^3 - 10^3] ...(i)

On differentiating Eq. (i) w.r.t. 't', we get

$\frac{dV}{dt} = \frac{4}{3}\pi$ (3(10 + x)^2) $\frac{dx}{dt}$ = -50 [given]

[- ve sign indicate that volume is decreasing as time passes]

$\Rightarrow$ 4π(10 + x)^2 $\frac{dx}{dt}$ = -50

At x = 5cm

$\frac{dx}{dt}$ [4π(10 + 5)^2] = -50

$\Rightarrow \frac{dx}{dt} = -\frac{50}{225(4\pi)} = -\frac{1}{9(2\pi)} = -\frac{1}{18\pi}$ cm/min

So, the thickness of the ice decreases at the rate of $\frac{1}{18\pi}$ cm/min

8.

(b) log 2

Explanation:

$2\int_0^1 \tan^{-1} x$ dx = $\int_0^1 \cot^{-1}$(1 - x + x^2)dx

= $\int_0^1 \left[\frac{\pi}{2} - \tan^{-1}\left(1 - x + x^2\right)\right]$ dx

$\Rightarrow \int_0^1 \tan^{-1}$(1 - x + x^2)dx

= $\frac{\pi}{2}\int_0^1 dx - 2\int_0^1 \tan^{-1} x$dx

= $\frac{\pi}{2} - 2\left[\left(\tan^{-1} x\right) x\right]_0^1$ + $2\int_0^1 \frac{1}{1+x^2} \cdot x$dx

= $\frac{\pi}{2} - 2\left(\frac{\pi}{4}\right) + \left[\log\left(1 + x^2\right)\right]_0^1$

= $\frac{\pi}{2} - \frac{\pi}{2}$ + log 2

= log 2

9.

(c) $\frac{13}{4}$

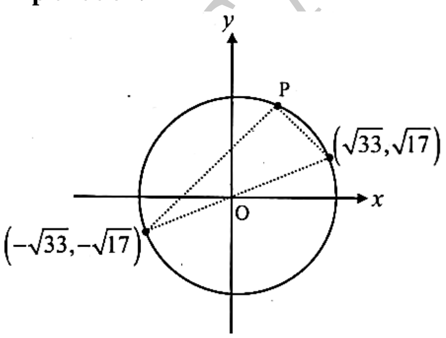
Explanation:

$\frac{13}{4}$

10.

(b) 3

Explanation:



Let P = (x, y)

P describes a circle x^2 + y^2 = 50

(x, y) = (1, 7), (5, 5), (7, 1) satisfy the condition.

11.

(b) y^2 = 2(x - 1)

Explanation:

We have

$y^2 = 4x$

We know that ends of focal chords are (at^2, 2at) and $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$

here a = 1

Let (h, k) be the mid point of the chord.

$\Rightarrow k = \frac{2t+\left(-\frac{2}{t}\right)}{2}$

$\Rightarrow 2k = 2t - \frac{2}{t}$

$\Rightarrow k = t - \frac{1}{t}$ ...(i)

h = $\frac{\mathbf{t}^2+\frac{1}{t^2}}{2}$

$\Rightarrow 2h = \left(t - \frac{1}{t}\right)^2$ + 2

$\Rightarrow$ 2h = k^2 + 2

To get equation of locus replace

h $\rightarrow$ x and k $\rightarrow$ y

2x = y^2 + 2

y^2 = 2(x − 1)

12.

(b) $\frac{3}{2g(e)}$

Explanation:

$\frac{3}{2g(e)}$

13.

(d) $x^2 + y^2 + z^2 = 2$

Explanation:

Let the moving point be (α, β, γ).

Given condition

$\Leftrightarrow \frac{(\alpha-2)^2}{1} + \frac{(\alpha+2)^2}{1} + \frac{(\beta-2)^2}{1} + \frac{(\beta+2)^2}{1} + \frac{(\gamma-2)^2}{1} + \frac{(\gamma+2)^2}{1}$ = 28

$\Rightarrow$ 2(α^2 + β^2 + γ^2) + 24 = 28

$\Rightarrow$ 2(α^2 + β^2 + γ^2) = 4

$\Rightarrow$ (α^2 + β^2 + γ^2) = 2

$\Rightarrow$ The locus of the point is $x^2 + y^2 + z^2 = 2$

14. **(a)** $3(\hat{i} + \hat{j} + \hat{k})$

Explanation:

Given, $\vec{a} \times \vec{b} = (\hat{i} - \hat{j})$

Taking cross product with $\vec{a}$

$\Rightarrow \vec{a} \times (\vec{a} \times \vec{b}) = \vec{a} \times (\hat{i} - \hat{j})$

By using $[\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\vec{c})\vec{b} - (\vec{a}\vec{b})\vec{c}]$

$\Rightarrow (\vec{a} \cdot \vec{b})\vec{a} - (\vec{a} \cdot \vec{a})\vec{b} = \hat{i} + \hat{j} + 2\hat{k}$

$\Rightarrow \vec{a} - 3\vec{b} = \hat{i} + \hat{j} + 2\hat{k} \Rightarrow 2\vec{a} - 6\vec{b} = 2\hat{i} + 2\hat{j} + 4\hat{k}$

$\Rightarrow \vec{a} - 6\vec{b} = 3\hat{i} + 3\hat{j} + 3\hat{k}$

15.

(b) 10

Explanation:

We have, Q_1 = 20 and Q_3 = 40

$\therefore$ Q.D. = $\frac{Q_3-Q_1}{2}$

= $\frac{40-20}{2}$

$= \frac{20}{2}$
= 10

16.
(d) $\frac{99}{200}$
Explanation:
If A draws 1, then B has options 2 to 100.
If A draws 2, then B has options 3 to 100 and so on.
Required probability = $\frac{1}{100} \times \frac{99}{100} + \frac{1}{100} \times \frac{98}{100} + \ldots + \frac{1}{100} \cdot \frac{1}{100}$
$= \frac{1}{100} \times \frac{99+98+\ldots+1}{100} = \frac{99}{200}$

17. **(a)** -2 sec α
Explanation:
$\sqrt{\frac{1-\sin\alpha}{1+\sin\alpha}} + \sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}}$
$= \frac{1-\sin\alpha+1+\sin\alpha}{\sqrt{1-\sin^2\alpha}}$
$= \frac{2}{|\cos\alpha|}$ = -2 sec α ...[$\because$ cos α < 0 in $\left(\frac{\pi}{2}, \pi\right)$]

18.
(c) 2
Explanation:
2

19.
(c) Not a well defined collection
Explanation:
Since, intelligency is not defined for students in a class i.e., Not a well defined collection.

20.
(c) $\frac{1}{8}$
Explanation:
Let A be the area of the triangle. Then
A = $\frac{1}{2}\begin{vmatrix} \frac{x_1}{a} & \frac{y_1}{a} & 1 \\ \frac{x_2}{b} & \frac{y_2}{b} & 1 \\ \frac{x_3}{c} & \frac{y_3}{c} & 1 \end{vmatrix} = \frac{1}{2abc}\begin{vmatrix} x_1 & y_1 & a \\ x_2 & y_2 & b \\ x_3 & y_3 & c \end{vmatrix}$
$= \frac{1}{4abc}\begin{vmatrix} x_1 & y_1 & 2a \\ x_2 & y_2 & 2b \\ x_3 & y_3 & 2c \end{vmatrix}$
$= \frac{1}{4abc}\begin{vmatrix} 2a & x_1 & y_1 \\ 2b & x_2 & y_2 \\ 2c & x_3 & y_3 \end{vmatrix}$
$= \frac{1}{4abc} \times \frac{abc}{2} = \frac{1}{8}$

MATHS (Section-B)

21. 22.0
Explanation:
Let f(x) = $ax^3 + bx^2 + cx + d$
$\Rightarrow$ f'(x) = $3ax^2 + 2bx + c \Rightarrow$ f''(x) = 6ax + 2b
$\Rightarrow$ f''(-1) = 0 $\Rightarrow$ -6a + 2b = 0 $\Rightarrow$ b = 3a
$\Rightarrow$ f'(1) = 0 $\Rightarrow$ 3a + 6a + c = 0 $\Rightarrow$ c = -9a
$\Rightarrow$ f(1) = -10 $\Rightarrow$ -5a + d = -10 ...(i)

$\Rightarrow$ f(-1) = 6 $\Rightarrow$ 11a + d = 6 ...(ii)

Subtract (ii) from (i),

We get a = 1, d = -5, b = 3, c = -9

Then f(x) = $x^3 + 3x^2 - 9x - 5$

So, f(3) = 27 + 27 - 27 - 5 = 22

22. 44.0

Explanation:

$I_1 : \vec{r} = (3\hat{i} - \hat{j} + 4\hat{k}) + t(\hat{i} + 2\hat{j} + 2\hat{k})$ and

$I_2 : \vec{r} = (3\hat{i} + 3\hat{j} + 2\hat{k}) + s(2\hat{i} + 2\hat{j} + \hat{k})$

$$\therefore \bar{b}_1 \times \bar{b}_2 = \begin{vmatrix} \hat{i}\hat{j}\hat{k} \\ 122 \\ 221 \end{vmatrix} = -2\hat{i} + 3\hat{j} - 2\hat{k}$$

So, $I : \vec{r} = \lambda(-2\hat{i} + 3\hat{j} - 2\hat{k})$

$\therefore$ Point of intersection of I and I_1 be P(2, -3, 2)

Let a point on I_2 be Q(2μ + 3, 2μ + 3, μ + 2)

$\because PQ = \sqrt{17}$

$\Rightarrow (2\mu + 1)^2 + (2\mu + 6)^2 + \mu^2 = 17$

$\Rightarrow \mu = -\frac{10}{9}$ or -2.

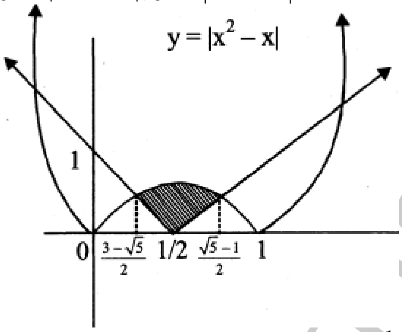
Given that, Q lies in 1st octant, then $Q\left(\frac{7}{9}, \frac{7}{9}, \frac{8}{9}\right)$

Hence, 18(a + b + c) = 44

23. 125.0

Explanation:

$y \geq |2x - 1|, y \leq |x^2 - x|$



Both curve are symmetric about x = $\frac{1}{2}$. Hence

$$A = 2\int_{\frac{3-\sqrt{5}}{2}}^{\frac{1}{2}} \left((x - x^2) - (1 - 2x)\right) dx$$

$$A = 2\int_{\frac{3-\sqrt{5}}{2}}^{\frac{1}{2}} \left(-x^2 + 3x - 1\right) dx = 2\left(\frac{-x^3}{3} + \frac{3}{2}x^2 - x\right)_{\frac{3-\sqrt{5}}{2}}^{\frac{1}{2}}$$

$A = \frac{-11+5\sqrt{5}}{6}$; Now, 6A + 11 = $5\sqrt{5}$ $\therefore$ $(6A + 11)^2 = 125$

24. 461.0

Explanation:

$S = \frac{b_1}{2} + \frac{b_2}{2^2} + \ldots + \frac{b_9}{2^9} + \frac{b_{10}}{2^{10}}$ ...(i)

$\Rightarrow \frac{S}{2} = \frac{b_1}{2^2} + \frac{b_2}{2^3} + \ldots \frac{b_9}{2^{10}} + \frac{b_{10}}{2^{11}}$ ...(ii)

subtracting equation (ii) from (i)

$\Rightarrow \frac{S}{2} = \frac{b_1}{2} + \left(\frac{a_1}{2^2} + \frac{a_2}{2^3} \ldots + \frac{a_9}{2^{10}}\right) - \frac{b_{10}}{2^{11}}$

$\Rightarrow \text{S} = b_1 - \frac{b_{10}}{1^{10}} + \left(\frac{a_1}{2} + \frac{a_2}{2^2} \ldots + \frac{a_9}{2^9}\right)$ ...(iii)

$\Rightarrow \frac{S}{2} = \frac{b_1}{2} - \frac{b_{10}}{2^{11}} + \left(\frac{a_1}{2^2} + \frac{a_2}{2^3} \ldots + \frac{a_9}{2^{10}}\right)$ ...(iv)

subtracting equation (iv) from (iii)

$\Rightarrow \frac{S}{2} = \frac{b_1}{2} - \frac{b_{10}}{2^{11}} + \left(\frac{a_1}{2} - \frac{a_9}{2^{10}}\right) + \left(\frac{1}{2^2} + \frac{2}{2^3} + \ldots + \frac{8}{2^9}\right)$

$\Rightarrow \frac{S}{2} = \frac{a_1+b_1}{2} - \frac{(b_{10}+2a_9)}{2^{11}} + \frac{T}{4}$

$\Rightarrow 2S = 2(a_1+b_1) - \frac{(b_{10}+2a_9)}{2^9} + T$

$\Rightarrow 2^7(2S-T) = 2^8(a_1+b_1) - \frac{(b_{10}+2a_9)}{4}$

Since, $a_n - a_{n-1} = n - 1$,

$\therefore \quad a_2 - a_1 = 1$

$a_3 - a_2 = 2$

$\vdots$

$a_9 - a_8 = 8$

$a_9 - a_1 = 1 + 2 + ... + 8 = 36$

$a_9 = 37 (a_1 = 1)$

Also, $b_n - b_{n-1} = a_{n-1}$

$\therefore b_{10} - b_1 = a_1 + a_2 + ... + a_9$

$= 1 + 2 + 4 + 7 + 11 + 16 + 22 + 29 + 37$

$\Rightarrow b_{10} = 130$ (As $b_1 = 1$)

$\therefore 2^7(2S - T) = 2^8(1 + 1) - (130 + 2 \times 37)$

$2^9 - \frac{204}{4} = 461$

25. 6

Explanation:

As (a, b, c) lies on 2x + y + z = 1 $\Rightarrow$ 2a + b + c = 1

$\Rightarrow$ 2a + 6a - 7a = 1

$\Rightarrow$ a = 1, b = 6, c = -7

$\therefore$ 7a + b + c = 7 + 6 - 7 = 6