Solution

MATHS

JEE main - Mathematics

MATHS (Section-A)

1.

(c) 8001

Explanation:

Given relations are

$\log_2(f(x)) = \log_2\left(2 + \frac{2}{3} + \frac{2}{9} \ldots \infty\right) \cdot \log_3\left(1 + \frac{f(x)}{f\left(\frac{1}{x}\right)}\right)$ and f(10) = 1001

when

$\log_2(f(x)) = \log_2\left[2\left(1 + \frac{1}{3} + \frac{1}{3^2} \ldots \infty\right)\right] \log_3\left(1 + \frac{f(x)}{f\left(\frac{1}{x}\right)}\right)$

$\Rightarrow \log_2(f(x)) = \log_2\left[2 \times \left(\frac{1}{1-\frac{1}{3}}\right)\right] \log_3\left(1 + \frac{f(x)}{f\left(\frac{1}{x}\right)}\right)$

$\Rightarrow \log_2(f(x)) = \log_2\left[2 \times \left(\frac{1}{\frac{2}{3}}\right)\right] \log_3\left(1 + \frac{f(x)}{f\left(\frac{1}{x}\right)}\right)$

$\Rightarrow \log_2(f(x)) = \log_2 3 \log_3\left(1 + \frac{f(x)}{f\left(\frac{1}{x}\right)}\right)$

$\Rightarrow \log_2(f(x)) = \log_2\left(1 + \frac{f(x)}{f\left(\frac{1}{x}\right)}\right)$

$\Rightarrow f(x) = 1 + \frac{f(x)}{f\left(\frac{1}{x}\right)}$

$\Rightarrow f(10) = 1 + \frac{f(10)}{f\left(\frac{1}{10}\right)}$

Now inserting f(10) = 1001

$\Rightarrow 1001 = 1 + \frac{1001}{f\left(\frac{1}{10}\right)}$

$\Rightarrow f\left(\frac{1}{10}\right) = \frac{1001}{1000} = 1 + \frac{1}{1000} = 1 + \left(\frac{1}{10}\right)^3$

Inserting $\frac{1}{10} = y$

f(y) = 1 + y^3

When y = 20

f(20) = 1 + 20^3 = 8001

2.

(c) $\frac{3}{4}$

Explanation:

Given, $\frac{2z-3i}{2z+i}$ is purely imaginary

$\Rightarrow \frac{2z-3i}{2z+i} + \frac{2\bar{z}+3i}{2\bar{z}-i} = 0$

$\Rightarrow 4x^2 + 4y^2 - 4y - 3 = 0$ ($\because$ z = x + iy)

Given that x + y^2 = 0

So, $y^4 + y^2 - y = \frac{3}{4}$

3.

(c) 680

Explanation:

$\sum_{r=1}^{15} r^2 \left(\frac{^{15}C_r}{^{15}C_{r-1}}\right)$

$\frac{^{15}C_r}{^{15}C_{r-1}} = \frac{\frac{15!}{(15-r)!r!}}{\frac{15!}{(r-1)!(16-r)!}} = \frac{\frac{1}{(15-r)!(r-1)!}}{\frac{1}{(r-1)!(15-r)!\cdot(16-r)}} = \frac{16-r}{r}$

$= \sum_{r=1}^{15} r^2 \left(\frac{16-r}{r}\right) = \sum_{r=1}^{15} r(16-r) = 16\sum_{r=1}^{15} r - \sum_{r=1}^{15} r^2$

$= \frac{16\times15\times16}{2} - \frac{15\times31\times16}{6}$

$= 8 \times 15 \times 16 - 5 \times 8 \times 31 = 1920 - 1240 = 680$

4. **(a)** 990

Explanation:

$(1 + x + x^2 + x^3)^{11} = (1 + x)^{11} (1 + x^2)^{11}$

x^4 appears as product of (x^2, x^2) or (x, x, x^2) or (x, x, x, x)

$\Rightarrow$ The coefficient of x^4

$= {}^{11}C_2 + {}^{11}C_2\, {}^{11}C_1 + {}^{11}C_4$

$= 55 + 11(55) + 330 = 990$

5.

(d) 7

Explanation:

7

6.

(b) $\frac{a^2b^2}{p^2}$

Explanation:

$p^2 = a^2 \cos^2\theta + b^2 \sin^2\theta$

Differentiating with respect to θ, we get

$2p\frac{dp}{d\theta} = 2(b^2 - a^2) \cos\theta \sin\theta = (b^2 - a^2) \sin 2\theta$

Differentiating with respect to 0, we get

$p\frac{d^2p}{d\theta^2} + \left(\frac{dp}{d\theta}\right)^2 = (a^2 - b^2)(\sin^2\theta - \cos^2\theta)$

$\Leftrightarrow p\frac{d^2p}{d\theta^2} + \frac{\left(\left(b^2-a^2\right)\sin\theta\cos\theta\right)^2}{p^2} = (a^2 - b^2)(\sin^2\theta - \cos^2\theta)$

$p^2 + p\frac{d^2p}{d\theta^2} = a^2\cos^2\theta + b^2\sin^2\theta + (a^2 - b^2)(\sin^2\theta - \cos^2\theta) - \frac{\left(b^2-a^2\right)^2\sin^2\theta\cos^2\theta}{a^2\cos^2\theta+b^2\sin^2\theta}$

$= a^2\sin^2\theta + b^2\cos^2\theta - \frac{\left(b^2-a^2\right)^2\sin^2\theta\cos^2\theta}{a^2\cos^2\theta+b^2\sin^2\theta}$

$= \frac{\left(a^4+b^4\right)\sin^2\theta\cos^2\theta+a^2b^2\left(\sin^4\theta+\cos^4\theta\right)-\left(a^4+b^4-2a^2b^2\right)\sin^2\theta\cos^2\theta}{p^2}$

$= \frac{a^2b^2\left(\sin^2\theta+\cos^2\theta\right)^2}{p^2}$

$= \frac{a^2b^2}{p^2}$

7. **(a)** $(e)^{\frac{1}{e}}$

Explanation:

Let $f(x) = (\frac{1}{x})^x \Leftrightarrow f(x) = x^{-x}$

Differentiating with respect to x, we get

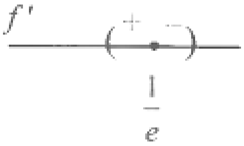
$f'(x) = -x^{-x}(1 + \log x)$

(using logarithmic differentiation)

f'{x) = 0

$\Leftrightarrow -x^{-x}(1 + \log x) = 0$

$\Rightarrow 1 + \log x = 0$
$\Leftrightarrow \log x = -1 \Rightarrow x = \frac{1}{e}$ (a point of relative maxima)



Maximum value of f at x = $\frac{1}{e}$ = f($\frac{1}{e}$) = $(e)^{\frac{1}{e}}$

8.
(b) 19
Explanation:
Given series can be define as $\lim\limits_{n \to \infty} \frac{3}{n} \sum_{r=0}^{n-1} \left(2 + \frac{r}{n}\right)^2$
$\frac{\mathrm{r}}{\mathrm{n}} \to \mathrm{x}, \frac{1}{\mathrm{n}} \to \mathrm{dx}$
$= 3 \int_0^1 (2 + x)^2 dx$ = 27 - 8 = 19

9. **(a)** sin B sin C
Explanation:
sin B sin C

10.
(b) 95
Explanation:
Tangent to the curve $x^2 = y - 6$ at (1, 7) is
x = $\frac{y+7}{2}$ - 6
$\Rightarrow$ 2x - y + 5 = 0 ......(i)
Equation of circle is $x^2 + y^2 + 16x + 12y + c = 0$
Centre (-8, -6)
r = $\sqrt{8^2 + 6^2 - c}$ = $\sqrt{100 - c}$
Since, line 2x - y + 5 = 0 also touches the circle.
$\therefore \sqrt{100 - c} = \left|\frac{2(-8) - (-6) + 5}{\sqrt{2^2 + 1^2}}\right|$
$\Rightarrow \sqrt{100 - c} = \left|\frac{-16 + 6 + 5}{\sqrt{5}}\right|$
$\Rightarrow \sqrt{100 - c} = |-\sqrt{5}|$
$\Rightarrow$ 100 - c = 5
$\Rightarrow$ c = 95

11.
(b) $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$
Explanation:
The line y = mx + c is tangent to the parabola
$y^2 = 4ar$
$\Rightarrow m^2x^2 + (2mc - 4a)x + c^2 = 0$ ...(i)
has equal roots
$\Rightarrow$ discriminant = 0
$\Rightarrow$ a = mc ...(ii)
Substitute in (i) to get $(mx - c)^2 = 0$
$\Rightarrow x = \frac{c}{m} = \frac{a}{m^2}$ ...[From (ii)]
Substitute in $y^2 = 4ax$ to get y = $\frac{2a}{m}$

12. **(a)** y = ce^{2x+x^3}

Explanation:

$\frac{dy}{dx}$ - $3x^2y$ = 2y ⇔ $\frac{1}{y}\left(\frac{dy}{dx}\right)$ = 2 + $3x^2$

⇔ $\frac{dy}{y}$ = (2 + $3x^2$) dx (Variables separable)

Integrating on both sides, we get

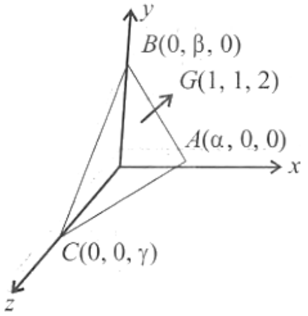
log y = 2x + x^3 + c_1

⇒ y = ce^{2x+x^3}, where c = e^{x_1}

13.

(c) $\frac{x-1}{2} = \frac{y-1}{2} = \frac{z-2}{1}$

Explanation:



∴ α = 3, β = 3 and γ = 6 as G is centroid.

∴ The equation of the plane is

$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1$

⇒ $\frac{x}{3} + \frac{y}{3} + \frac{z}{6} = 1$ ⇒ 2x + 2y + z = 6

∴ The required line is, $\frac{x-1}{2} = \frac{y-1}{2} = \frac{z-2}{1}$

14. **(a)** 1

Explanation:

The projection of $\alpha\hat{i} + \beta\hat{j}$ on $3\hat{i} - 4\hat{j} = 1$

⇔ $\frac{(\alpha\hat{i}+\beta\hat{j})\cdot(3\hat{i}-4\hat{j})}{|3\hat{i}-4\hat{j}|} = 1$

⇔ $\frac{3\alpha-4\beta}{5} = 1$

⇔ $3\alpha - 4\beta = 5$

⇒ 3(1 + 2β) - 4β = 5 ...[$\alpha - 2\beta$ = 1]

⇒ 2β = 2 ⇔ β = 1

15. **(a)** 22

Explanation:

Let x_1, x_2, ... , x_{11} be the 11 observations. Then, $\frac{x_1+x_2+\ldots+x_{11}}{11}$ = 70

⇔ x_1 + x_2 + ... + x_{11} = 770 ...(i)

Given, $\frac{x_1+x_2+\ldots+x_6}{6}$ = 72

⇒ x_1 + x_2 + ... + x_6 = 432 ...(ii)

and $\frac{x_6+x_7+\ldots+x_{11}}{6}$ = 60

⇒ x_6 + x_7 + ... + x_{11} = 360 ...(iii)

(ii) + (iii) - (i)

⇒ (x_1 + x_2 + ... + x_6) + (x_6 + x_7 + ... + x_{11}) - (x_1 + x_2 + ... + x_{11})

= 432 + 360 - 770

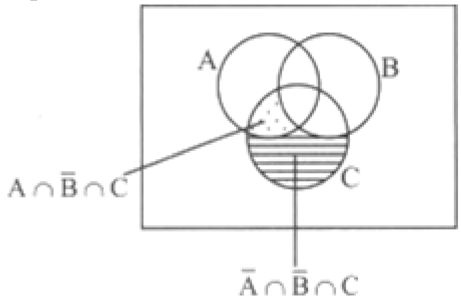
⇒ x_6 = 22

⇒ Middlemost observation = 22

16.

(b) $\frac{17}{42}$

Explanation:



Here P(C) = $P(B \cap C)+P(A \cap \overline{B} \cap C)+P(\overline{A} \cap \overline{B} \cap C)$

$\Rightarrow \frac{5}{6}=P(B \cap C)+\frac{2}{7}+\frac{1}{7}$

$\Leftrightarrow P(B \cap C)=\frac{5}{6}-\frac{3}{7}=\frac{35-18}{42}=\frac{17}{42}$

17. **(a)** 0

Explanation:

0

18.

(d) $\frac{2}{5}(2x - y + 4)^2 - \frac{3}{5}(x + 2y - 3)^2 = 1$

Explanation:

Given, 2a = $\sqrt{2}$

$\Rightarrow$ a = $\frac{1}{\sqrt{2}}$

Also, 2b = $\frac{2}{\sqrt{3}}$

$\Rightarrow$ b = $\frac{1}{\sqrt{3}}$

If we take the two axes as the new coordinate system, and the point of intersection of the axes as the new origin, then in the new coordinate system, equation of the hyperbola will be:

$\frac{\mathbf{X}^2}{\mathbf{a}^2}-\frac{\mathbf{Y}^2}{\mathbf{b}^2}$ = 1

$\Rightarrow 2X^2 - 3Y^2 = 1$

Let P(x, y) be the coordinates of a point on the hyperbola in original x-y system, then

X = $\frac{|2x-y+4|}{\sqrt{5}}$, Y = $\frac{|\mathbf{x}+\mathbf{2y}-\mathbf{3}|}{\sqrt{\mathbf{5}}}$ ($\because$ X is the distance of a point on hyperbola from 2x - y + 4 = 0 and Y is the distance of a point on hyperbola from x + 2y - 3 = 0)

So, the required equation is

$\frac{2(2x-y+4)^2}{5}-\frac{3(x+2y-3)^2}{5}$ = 1

19.

(b) $A \cap B \neq \phi$

Explanation:

$\because$ y = e^x, y = e^{-x} will meet, when $e^x = e^{-x}$

$\Rightarrow e^{2x} = 1$,

$\therefore$ x = 0, y = 1

A and B meet on (0, 1), A ∩ B = ϕ

20.

(c) 1

Explanation:

1

MATHS (Section-B)

21. 2

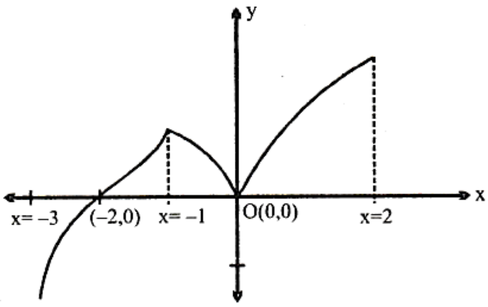
Explanation:

Clearly, $g'(x) = f(x) = \begin{cases} (x+2)^3, & -3 < x \le -1 \\ x^{\frac{2}{3}}, & -1 < x < 2 \end{cases}$

$\Rightarrow g'(x)$ is a continuous funtion

Sign scheme of g''(x)

$\Rightarrow g'(x)$ has a local maximum at x = -1 and local minimum at x = 0



Graph of g'(x) = f(x)

Hence, $g'(x)$ has two extremum points.

22. 108

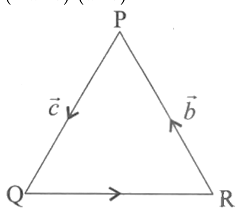
Explanation:

$\because \vec{a} + \vec{b} + \vec{c} = \vec{0}$

$\Rightarrow \vec{c} = -\vec{a} - \vec{b}$

Now, $\frac{\vec{a}\cdot(\vec{c}-\vec{b})}{\vec{c}\cdot(\vec{a}-\vec{b})} = \frac{|\vec{a}|}{|\vec{a}|+|\vec{b}|}$

$\frac{\vec{a}\cdot(-\vec{a}-2\vec{b})}{(-\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})} = \frac{3}{7}$



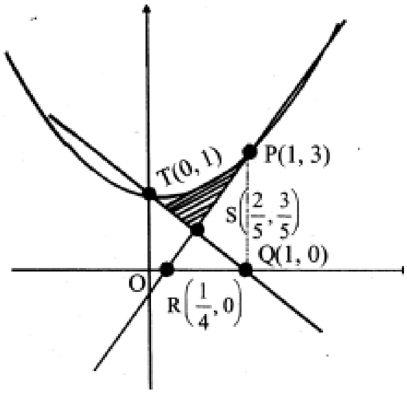
$\Rightarrow \frac{9+2\vec{a}\cdot\vec{b}}{9-16} = \frac{3}{7}$

$\Rightarrow \vec{a}\cdot\vec{b}$ = -6

$\Rightarrow |\vec{a}\times\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2$ = 9 $\times$ 16 - 36 = 108

23. 16.0

Explanation:



y = 2x^2 + 1 $\Rightarrow$ $\frac{dy}{dx}$ = 4x

Tangent at (1, 3), y - 3 = $\left.\frac{dy}{dx}\right|_{\substack{x=1\\y=3}}(x-1)$

$\Rightarrow$ y = 4 x - 1

Now, A = $\int_0^1 \left(2x^2+1\right)dx$ - area of ($\triangle$QOT) - area of($\triangle$PQR) + area of ($\triangle$QRS)

$A=\left(\frac{2}{3}+1\right)-\frac{1}{2}-\frac{9}{8}+\frac{9}{40}=\frac{16}{60}$

$\Rightarrow$ 60A = 16

24. 16.0

Explanation:

$$S=\frac{a_1}{2}+\frac{a_2}{2^2}+\frac{a_3}{2^3}+....$$

$$\frac{S}{2}=\frac{a_1}{2^2}+\frac{a_2}{2^3}+....$$

$$\frac{S}{2}=\frac{a_1}{2}+d\left(\frac{1}{2^2}+\frac{1}{2^3}+...\right)$$

$$\frac{S}{2}=\frac{a_1}{2}+d\left(\frac{\frac{1}{4}}{1-\frac{1}{2}}\right)$$

$\therefore$ S = a_1 + d = a_2 = 4 or $4a_2$ = 16

25. 10

Explanation:

$A^2 = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} x^2+1 & x \\ x & 1 \end{bmatrix}$

$A^4 = \begin{bmatrix} x^2+1 & x \\ x & 1 \end{bmatrix}\begin{bmatrix} x^2+1 & x \\ x & 1 \end{bmatrix}$

$= \begin{bmatrix} \left(x^2+1\right)^2+x^2 & x\left(x^2+1\right)+x \\ x\left(x^2+1\right)+x & x^2+1 \end{bmatrix}$

Given that $(x^2 + 1)^2 + x^2 = 109$

$x^4 + 3x^2 - 108 = 0$

$\Rightarrow (x^2 + 12)(x^2 - 9) = 0$

$\therefore x^2 = 9$

$a_{22} = x^2 + 1 = 9 + 1 = 10$