Solution

MATHEMATICS

Class 12 - Mathematics

Section A

1.

(b) $a_{ij} = b_{ij}$

Explanation:

Two matrices A = $[a_{ij}]$ and B = $[b_{ij}]$ are said to be equal, if

i. they are of the same order.
ii. each element of A is equal to the corresponding element of B i.e. $a_{ij} = b_{ij}$ by for all i and j.

2.

(c) unique solution

Explanation:

The given system of linear equations is:-

$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$
$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$
$-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Let x^2 = p, y^2 = q, x^2 = r

The equation becomes,

$\frac{p}{a^2}+\frac{q}{b^2}-\frac{r}{c^2}=1$
$\frac{p}{a^2}-\frac{q}{b^2}+\frac{r}{c^2}=1$
$-\frac{p}{a^2}+\frac{q}{b^2}+\frac{r}{c^2}=1$

The matrix equation corresponding to the above system of equation is

$$\begin{bmatrix}\frac{1}{a^2} & \frac{1}{b^2} & -\frac{1}{c^2}\\ \frac{1}{a^2} & \frac{-1}{b^2} & \frac{1}{c^2}\\ -\frac{1}{a^2} & \frac{1}{b^2} & \frac{1}{c^2}\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}5\\9\\\mu\end{bmatrix}$$

Let A = $\begin{bmatrix}\frac{1}{a^2} & \frac{1}{b^2} & -\frac{1}{c^2}\\ \frac{1}{a^2} & \frac{-1}{b^2} & \frac{1}{c^2}\\ -\frac{1}{a^2} & \frac{1}{b^2} & \frac{1}{c^2}\end{bmatrix}$

$\therefore$ |A| = $\begin{vmatrix}\frac{1}{\mathrm{a}^2} & \frac{1}{\mathrm{b}^2} & -\frac{1}{\mathrm{c}^2}\\ \frac{1}{\mathrm{a}^2} & \frac{-1}{\mathrm{b}^2} & \frac{1}{\mathrm{c}^2}\\ -\frac{1}{\mathrm{a}^2} & \frac{1}{\mathrm{b}^2} & \frac{1}{\mathrm{c}^2}\end{vmatrix}$

Taking common $\frac{1}{a^2}$ from C_1, $\frac{1}{b^2}$ from C_2 and $\frac{1}{c^2}$ from C_3 we get,

|A| = $\frac{1}{a^2}\frac{1}{b^2}\frac{1}{c^2}\begin{vmatrix}1 & 1 & -1\\ 1 & -1 & 1\\ -1 & 1 & 1\end{vmatrix}=\frac{-4}{a^2b^2c^2}\neq 0$.

3.

(c) a unique solution

Explanation:

a unique solution

The given system of equations can be written in matrix form as follows:

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -18 \end{bmatrix}$$

AX = B

Here,

A = $\begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and B = $\begin{bmatrix} 2 \\ 6 \\ -18 \end{bmatrix}$

|A| = 1(- 1 - 2) - 1 (-3 - 6) + 1(3 + 3)

= -3 + 3 + 6

= 6 not equal to 0.

So, the given system of equations has a unique solution.

4.

(c) $2\sqrt{1-x^2}$

Explanation:

$y = x\sqrt{1-x^2} + \sin^{-1}(x)$

$\Rightarrow \frac{dy}{dx} = x\left\{\frac{1}{2\sqrt{1-x^2}}(-2x)\right\} + \sqrt{1-x^2} + \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \frac{dy}{dx} = \frac{-x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2} + \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \frac{dy}{dx} = \frac{-x^2+1-x^2+1}{\sqrt{1-x^2}}$

$\Rightarrow \frac{dy}{dx} = \frac{-2x^2+2}{\sqrt{1-x^2}}$

$\Rightarrow \frac{dy}{dx} = 2\sqrt{1-x^2}$

5.

(c) (- 1 , 2 , - 2)

Explanation:

(- 1 , 2 , - 2)

Direction ratios of OP = Coordinates of P - Coordinates of O

-1, 2, 2 = (x - 0), (y - 0), (z - 0)

Thus, coordinates of P are (-1,2,-2)

6.

(d) 5

Explanation:

Here we see that, there seem to be 3 constants in the term $(c_3 + c_4)\, a^{x+c_5}$, which actually is a single constant. So, c3, c4, and c5 are combined to give a single constant.

So, in totality there are 5 constants. Hence, the order of the equation is 5.

7.

(c) $x_1 = 2$, $x_2 = 6$, Z = 36

Explanation:

We need to maximize the function z = $3x_1 + 5x_2$

First, we will convert the given inequations into equations, we obtain the following equations: $3x_1 + 2x_2 = 18$, $x_1 = 4$, $x_2 = 6$,

$x_1 = 0$ and $x_2 = 0$

Region represented by $3x_1 + 2x_2 \leq 18$

The line $3x_1 + 2x_2 = 18$ meets the coordinate axes at A(6, 0) and B(0, 9) respectively. By joining these points we obtain the line

$3x_1$ +

$2x_2 = 18$

Clearly (0, 0) satisfies the inequation $3x_1 + 2x_2 = 18$. So, the region in the plane which contain the origin represents the solution set of the inequation $3x_1 + 2x_2 = 18$

Region represented by $x_1 \leq 4$:

The line $x_1 = 4$ is the line that passes through C(4, 0) and is parallel to the Y axis. The region to the left of the line $x_1 = 4$ will satisfy the inequation $x_1 \leq 4$

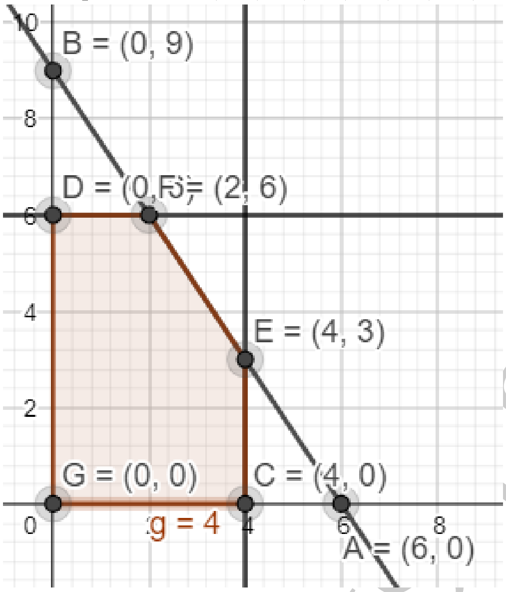
Region represented by $x_2 \leq 6$ The line $x_2 = 6$ is the line that passes through D(0, 6) and is parallel to the x axis. The region below the line $x_2 = 6$ will satisfy the inequation $x_2 \leq 6$.

Region represented by $x_1 \geq 0$ and $x_2 \geq 0$

since, every point in the first quadrant satisfies these inequations. So, the first quadrant is the region represented by the inequations $x_1 \geq$ oand $x_2 \geq 0$

The feasible region determined by the system of constraints, $3x_1 + 2x_2 \leq 18$, $x_1 \leq 4$, $x_2 \leq 6$, $x_1 \geq 0$, and $x_2 \geq 0$, are as follows :

Corner points are O(0, 0), D(0, 6), F(2, 6), E(4, 3) and C(4, 0).



The values of the objective function at these points are given in the following table

Points : Value of Z

O(0, 0) : 3(0) + 5(0) = 0

D(0, 6) : 3(0) + 5(6) = 30

F(2, 6) : 3(2) + 5(6) = 36

E(4, 3) : 3(4) + 5(3) = 27

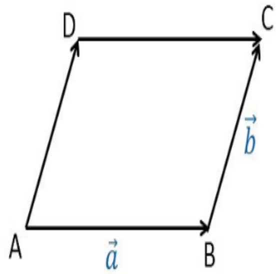
C(4, 0) : 3(4) + 5(0) = 12

We see that the maximum value of the objective function Z is 36 which is at F(2, 6)

8.

(b) $\sqrt{42}$

Explanation:



Given that,

$a = 3\hat{\imath} + 1\hat{\jmath} + 4\hat{k}$

$b = 1\hat{\imath} - 1\hat{\jmath} + 1\hat{k}$

Area of Parallelogram ABCD $= |\vec{a} \times \vec{b}|$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix}$$

$= \hat{i}(1 \times 1 - (-1) \times 4) - \hat{j}(3 \times 1 - 1 \times 4) + \hat{k}(3 \times -1 - 1 \times 1)$

$= 5\hat{i} + \hat{j} - 4\hat{k}$

$|\vec{a} \times \vec{b}| = \sqrt{25 + 1 + 16} = \sqrt{42}$

9.

(c) sin(log x)+C

Explanation:

Put log x = t , we get ;

$\int \frac{\cos(\log x)}{x} dx = \int \cos t dt = \sin t = \sin(\log x) + C$

10. **(a)** A + B

Explanation:

AB = B ⇒ (AB)A = BA

⇒ A(BA) = BA ⇒ A(A) = A,

⇒ A^2 = A

AB = B ⇒ B(AB) = BB

⇒ (BA)B = B^2

⇒ AB = B^2

⇒ B = B^2

∴ $A^2 + B^2$ = A + B

11. **(a)** Z has no maximum value

Explanation:

Objective function is Z = - x + 2 y ……………………(1).

The given constraints are : x ≥ 3, x + y ≥ 5, x + 2y ≥ 6, y ≥ 0.

Corner points	Z = - x + 2y
D(6,0)	-6
A(4,1)	-2
B(3,2)	1

Here, the open half plane has points in common with the feasible region.

Therefore, Z has no maximum value.

12.

(c) $\pm \frac{\hat{i}+\hat{j}}{\sqrt{2}}$

Explanation:

Since, unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ = $\pm \frac{a \times b}{|a \times b|}$

$$\therefore \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix}$$

$= \hat{i}[1 + 1] - \hat{j}[-1 - 1] + \hat{qk}[1 - 1]$

$= 2\hat{i} + 2\hat{j} + 0\hat{k} = 2(\hat{i} + \hat{j})$

and $|\vec{a} \times \vec{b}| = \sqrt{4 + 4} = 2\sqrt{2}$

∴ Required unit vector

$= \pm \frac{2(\hat{\mathbf{i}}+\hat{\mathbf{j}})}{2\sqrt{2}} = \pm \frac{\hat{\mathbf{i}}+\hat{\mathbf{j}}}{\sqrt{2}}$

13.

(c) $\frac{1}{2}$

Explanation:

$\frac{1}{2}$

14. **(a)** $\frac{1-\mathrm{P}(\mathrm{A} \cup \mathrm{B})}{\mathrm{P}\left(\mathrm{B}^{\prime}\right)}$

Explanation:

$\because P(A)>0$ and $P(B) \neq 1$

$P\left(A^{\prime} / B^{\prime}\right)=\frac{P\left(A^{\prime} \cap B^{\prime}\right)}{P\left(B^{\prime}\right)}=\frac{1-P(A \cup B)}{P\left(B^{\prime}\right)}$

15.

(c) xy = C

Explanation:

xy = C

16.

(d) $\frac{19}{9}$

Explanation:

Scalar projection of $\vec{a}$ on $\vec{b}$ = $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

= $\frac{(\hat{i}-2 \hat{j}+\hat{k}) \cdot(4 \hat{i}-4 \hat{j}+7 \hat{k})}{|4 i-4 \hat{j}+7 \hat{k}|}$

$=\frac{(4+8+7)}{\sqrt{(4)^{2}+(-4)^{2}+(7)^{2}}}=\frac{19}{9}$

which is the required scalar projection of $\vec{a}$ and $\vec{b}$.

17.

(c) – 1

Explanation:

$\frac{d y}{d z}=\frac{\frac{d}{d x}\left(\tan ^{-1} x\right)}{\frac{d}{d x}\left(\cot ^{-1} x\right)}=\frac{\frac{1}{1+x^{2}}}{-\frac{1}{1+x^{2}}}=-1$

18.

(c) $\frac{3 \sqrt{2}}{2}$

Explanation:

On comparing the given equations with: $\overrightarrow{r}=\overrightarrow{a_{1}}+\lambda \overrightarrow{b_{1}}$, and $\overrightarrow{r}=\overrightarrow{a_{2}}+\mu \overrightarrow{b_{2}}$, we get:

$\overrightarrow{a_{1}}=\hat{i}+2 \hat{j}+\hat{k}, \overrightarrow{b_{1}}=\hat{i}-\hat{j}+\hat{k}$, and $\overrightarrow{a_{2}}=2 \hat{i}-\hat{j}-\hat{k}, \overrightarrow{b_{2}}=2 \hat{i}+\hat{j}-2 \hat{k}$

$\therefore S . D .=\left|\frac{(\overrightarrow{b_{1}} \times \overrightarrow{b_{2}}) \cdot(\overrightarrow{a_{2}}-\overrightarrow{a_{1}})}{\left|\overrightarrow{b_{1}} \times \overrightarrow{b_{2}}\right|}\right|=\left|\frac{(-3 \hat{i}+3 \hat{k}) \cdot(\hat{i}-3 \hat{j}-2 \hat{k})}{3 \sqrt{2}}\right|$

$=\left|\frac{-3+-6}{3 \sqrt{2}}\right|=\left|\frac{-9}{3 \sqrt{2}}\right|=\frac{3}{\sqrt{2}}=\frac{3 \sqrt{2}}{2}$

19. **(a)** Both A and R are true and R is the correct explanation of A.

Explanation:

Circumference of circle with radius r is given by C = 2πr

Differentiating w.r.t. 't', we get

$\frac{d C}{d t}=2 \pi \frac{d r}{d t}$

Given, $\frac{d C}{d t}=10 \mathrm{~cm} / \mathrm{s}$

$\therefore 10=2 \pi \frac{d r}{d t}$

$\Rightarrow \frac{d r}{d t}=\frac{5}{\pi} \mathrm{cm} / \mathrm{s}$

Now, Area of circle, A = πr^2

$\therefore \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

Substituting r = 3 cm and $\frac{dr}{dt} = \frac{5}{\pi}\text{cm/s}$, we get

$\frac{dA}{dt} = 2\pi \times 3 \times \frac{5}{\pi}$

$\therefore \frac{dA}{dt} = 30\ \text{cm}^2/\text{s}$

20. **(a)** Both A and R are true and R is the correct explanation of A.

Explanation:

By definition, a Relation in R us to be refelexive if X R X, $\forall$ x $\in$ Z

So R is true.

a - a = 0 $\Rightarrow$ 2 divides a - a $\Rightarrow$ a R a

Hence, R is reflexive and A is true.

Section B

21. $\tan^{-1}\left(\tan \frac{3\pi}{\pi}\right) \neq \frac{3\pi}{4}$ as $\frac{3\pi}{4} \notin \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

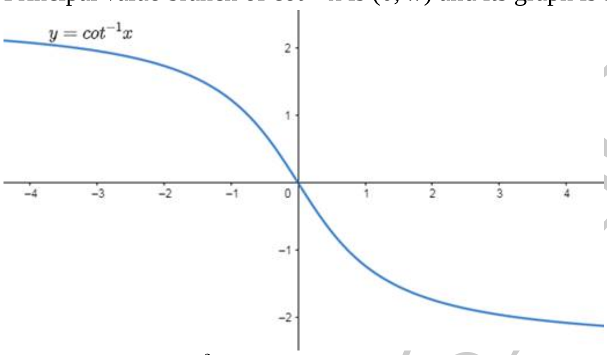
$\because \tan^{-1}\left(\tan \frac{3\pi}{4}\right) = \tan^{-1}\left[\tan\left(\pi - \frac{\pi}{4}\right)\right]$

$= \tan^{-1}\left[-\tan\left(\frac{\pi}{4}\right)\right]$

$= -\frac{\pi}{4}$

OR

Principal value branch of $\cot^{-1} x$ is (0, π) and its graph is shown below.



22. Given: f(x) = x + $\frac{a^2}{x}$

$\because$ f'(x) = 1 - $\frac{a^2}{x^2}$

f''(x) = $\frac{2a^2}{x^3}$

For maxima and minima, we must have

f'(x) = 0

$\Rightarrow 1 - \frac{a^2}{x^2} = 0$

$\Rightarrow$ x^2 - a^2 = 0

$\Rightarrow x = \pm a$

Now,

f''(a) = $\frac{2}{a}$ > 0 as a > 0

$\because$ x = a is point of minima

f''(-a) = $\frac{-2}{a}$ < 0 as a > 0

$\because$ x = -a is point of maxima

Hence,

Local max value = f(-a) = -2a

Local min value = f(a) = 2a.

23. Given, P(x) = $0.005x^3 + 0.02x^2 + 30x$

On differentiating both sides w.r.t. x, we get

$P'(x) = 3 \times 0.005x^2 + 2 \times 0.02x + 30$

On putting x = 3 we get

$P'(3) = 3 \times 0.005 \times 9 + 2 \times 0.02 \times 3 + 30$

= 0.135+ 0.12 + 30

= 30.255

Therefore, the marginal increase in pollution content when 3 diesel vehicles are added is 30.255.

Pollution content in the city increases with the increase in number of diesel vehicles.

OR

Here we have

f(x) = $\frac{4}{x+2} + x$

∵ f'(x) $\frac{-4}{(x+2)^2}$ + 1

f''(x) = $\frac{8}{(x+2)^3}$

For maximum and minimum value,

f'(x) = 0

⇒ $\frac{-4}{(x+2)^2}$ + 1 = 0

⇒ $(x + 2)^2 = 4$

⇒ $x^2 + 4x = 0$

⇒ x (x + 4) = 0

x = 0, -4

Now, f''(0) = 1 > 0

∵ x = 0 is point of minima

f''(-4) = -1 < 0

∵ local max value = f(- 4) = -6

local min value = f (0) = 2.

24. Let I $= \int_0^1 \frac{x}{x+1} dx$ .Then we have

$I = \int_0^1 1 - \frac{1}{x+1} dx$

$\Rightarrow I = [x - \log(x+1)]_0^1$

$\Rightarrow I = 1 - \log 2 - (0 - \log 1)$

$\Rightarrow I = \log e - \log 2$

$\Rightarrow I = \log \frac{e}{2}$

25. Given: f(x) = sin x

Theorem:- Let f be a differentiable real function defined on an open interval (a,b).

i. If f'(x) > 0 for all $x \in (a, b)$, then f(x) is increasing on (a, b)
ii. If f'(x) < 0 for all ,$x \in (a, b)$ then f(x) is decreasing on (a, b)

For the value of x obtained in (ii) f(x) is increasing and for remaining points in its domain it is decreasing.

Now, we have,

f(x) = sin x

$\Rightarrow f(x) = \frac{d}{dx}(\sin x)$

Now, as given

$x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

That is 4^{th} quadrant, where

= cosx > 0

= f'(x) > 0

hence, it is the condition for f(x) to be increasing

Thus the function f(x) is increasing on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Section C

26. Let I = $\int x . cos^{-1} x$

Now, integrating by parts, we get,

$I = \cos^{-1} x \int x dx - \int \left\{ \left(\frac{d}{dx} \cos^{-1} x \right) \int x dx \right\} dx$

$= \cos^{-1} x \cdot \frac{x^2}{2} - \int \frac{-1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} dx$

$= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} \int \frac{1-x^2-1}{\sqrt{1-x^2}} dx$

$= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} \int \left\{ \sqrt{1-x^2} + \left(\frac{-1}{\sqrt{1-x^2}} \right) \right\} dx$

$= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} \int \sqrt{1-x^2} dx - \frac{1}{2} \int \left(\frac{-1}{\sqrt{1-x^2}} \right) dx$

$= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} I_1 - \frac{1}{2} \cos^{-1} x$ ...(i)

Now, $I_1 = \int \sqrt{1-x^2} dx$

$I_1 = x\sqrt{1-x^2} - \int \frac{d}{dx} \sqrt{1-x^2} \int dx$

$I_1 = x\sqrt{1-x^2} - \int \frac{-2x}{2\sqrt{1-x^2}} x \cdot dx$

$= x\sqrt{1-x^2} - \int \frac{-x^2}{\sqrt{1-x^2}} dx$

$= x\sqrt{1-x^2} - \int \frac{1-x^2-1}{\sqrt{1-x^2}} dx$

$= x\sqrt{1-x^2} - \left\{ \int \sqrt{1-x^2} dx + \int \frac{-dx}{\sqrt{1-x^2}} \right\}$

$\therefore I_1 = x\sqrt{1-x^2} - \{I_1 + \cos^{-1} x\}$

$\Rightarrow 2I_1 = x\sqrt{1-x^2} - \cos^{-1} x$

$\Rightarrow I_1 = \frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\cos^{-1} x$

Now, substituting in (i), we get,

$I = \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} \left(\frac{x}{2} \sqrt{1-x^2} - \frac{1}{2}\cos^{-1} x \right) - \frac{1}{2}\cos^{-1} x$

$= \frac{(2x^2-1)}{4} \cos^{-1} x - \frac{x}{4}\sqrt{1-x^2} + C$

27. Let A, E_1 and E_2 denote the events that the vehicle meets the accident, is a scooter and is a motorcycle, respectively.

Therefore,we have,

$P(E_1) = \frac{2000}{5000} = 0.4$

$P(E_2) = \frac{3000}{5000} = 0.6$

Now,we have,

$P(\frac{A}{E_1}) = 0.01$

$P(\frac{A}{E_2}) = 0.02$

Using Bayes' theorem, we have,

Required probability is given by, $P(E_2/A) = \frac{P(E_2)P(A/E_2)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)}$

$= \frac{0.6 \times 0.02}{0.6 \times 0.02 + 0.4 \times 0.01}$

$= \frac{0.012}{0.012+0.004} = \frac{0.012}{0.016} = \frac{3}{4}$

28. To find: $\int \frac{x^2 dx}{(a^6 - x^6)}$

Formula to be Used in this: $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log\left| \frac{a+x}{a-x} \right| + C$

Let y = x^3 ...(i)

Differentiating both sides, we get

dy = $3x^2$ dx

Substituting in given equations,

$\Rightarrow \int \frac{\frac{1}{3} dy}{a^6 - y^2}$

$\Rightarrow \frac{1}{3} \int \frac{1}{(a^3)^2 - y^2} dy$

$\Rightarrow \frac{1}{3} \times \frac{1}{2a^3} \times \log\left| \frac{a^3+y}{a^3-y} \right| + c$

$\Rightarrow \frac{1}{6a^3} \log\left| \frac{a^3+y}{a^3-y} \right| + C$

From (1),

$\Rightarrow \frac{1}{6a^3} \log\left| \frac{a^3+x^3}{a^3-x^3} \right| + C$

Therefore, we have ..

$\int \frac{x^2 dx}{(a^6-x^6)} = \frac{1}{6a^3} \log\left| \frac{a^3+x^3}{a^3-x^3} \right| + C$

OR

Let I = $\int \frac{4x^4+3}{(x^2+2)(x^2+3)(x^2+4)} dx$

Let $x^2 = y$

$\because \frac{4x^4+3}{(x^2+2)(x^2+3)(x^2+4)} = \frac{4y^2+3}{(y+2)(y+3)(y+4)}$

Now by using partial fraction.

Let $\frac{4y^2+3}{(y+2)(y+3)(y+4)} = \frac{4y^2+3}{(y+2)(y+3)(y+4)}$

$\Rightarrow 4y^2 + 3 = A(y + 3)(y + 4) + B(y + 2)(y + 4) + C(y + 2)(y + 3)$

For y = -2, A = $\frac{19}{2}$

For y = -3, B = -39

For y = -4,C = $\frac{67}{2}$

Thus, I = $\frac{19}{2}\int \frac{dx}{x^2+2} - 39\int \frac{dx}{x^2+3} + \frac{67}{2}\int \frac{dx}{x^2+4}$

$\Rightarrow$ I = $\frac{19}{2\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) - \frac{39}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + \frac{67}{4}\tan^{-1}\left(\frac{x}{2}\right) + c$

29. It is given that (x - y)(dx + dy) = dx - dy

$\Rightarrow$ (x - y + 1)dy = (1 - x + y)dx

$\Rightarrow \frac{dy}{dx} = \frac{1-x+y}{x-y+1}$ ......(i)

Let x – y = t

$\Rightarrow \frac{d}{dx}(x-y) = \frac{dt}{dx}$

$\Rightarrow 1 - \frac{dy}{dx} = \frac{dt}{dx}$

Now, let us substitute the value of x-y and $\frac{dy}{dx}$ in equation (i), we get,

$1 - \frac{dt}{dx} = \frac{1-t}{1+t}$

$\Rightarrow \frac{dt}{dx} = 1 - \left(\frac{1-t}{1+t}\right)$

$\Rightarrow \frac{dt}{dx} = \frac{(1+t)-(1-t)}{1+t}$

$\Rightarrow \frac{dt}{dx} = \frac{2t}{1+t}$

$\Rightarrow \left(\frac{1+t}{t}\right) dt = 2dx$

$\Rightarrow \left(1 + \frac{1}{t}\right) dt = 2dx$ .....(ii)

On integrating both side, we get,

t + log|t| = 2x + C

$\Rightarrow$ (x - y) + log |x - y| = 2x + C

$\Rightarrow$ log|x – y| = x + y + C ....(iii)

Now, y = -1 at x = 0

Then, equation (iii), we get,

log 1 = 0 - 1 + C

$\Rightarrow$ C = 1

Substituting C = 1 in equation (iii), we get,

log|x - y| = x + y + 1

Therefore, a particular solution of the given differential equation is log|x - y| = x + y + 1.

OR

We have, (x+1) $\frac{dy}{dx}$ = $2e^{-y}$ -1, y(0) = 0

$\Rightarrow$ (x + 1) dy = ($2e^{-y}$ - 1) dx

$\Rightarrow \frac{1}{x+1}dx = \frac{1}{2e^{-y}-1}dy$ [On separating the variables]

Integrating both sides,

$\Rightarrow \int \frac{1}{x+1}dx = \int \frac{1}{2e^{-y}-1}dy$

$\Rightarrow \int \frac{1}{x+1}dx = \int \frac{e^y}{2-e^y}dy$

$\Rightarrow \int \frac{1}{x+1}dx = -\int \frac{e^y}{e^y-2}dy$

$\Rightarrow$ log |x + 1| = - log |e^y - 2| + log C

$\Rightarrow$ log |x + 1| + log |e^y - 2| + log C

$\Rightarrow$ log |(x + 1) (e^y - 2)| = log C

$\Rightarrow |(x + 1) (e^y - 2)| = C$ ...(i)

It is given that y (0) = 0 i.e. y = 0. Putting x = 0 and y = 0 in (i), we get

$|(0 + 1) (1 - 2)| = C \Rightarrow C = 1$

Putting C = 1 in (i), we get

$|(x + 1) (e^y - 2) = 1$

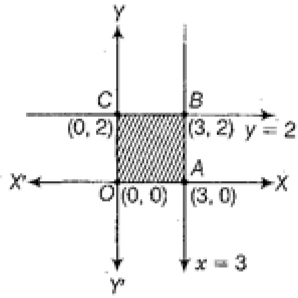
$\Rightarrow (x + 1) (e^y - 2) = \pm 1$

$\Rightarrow e^y - 2 = -\frac{1}{x+1}$

$\Rightarrow e^y = \left(2 - \frac{1}{x+1}\right)$

$\Rightarrow y = \log\left(2 - \frac{1}{x+1}\right)$, which is required solution.

30. Maximise Z = 11x + 7y, subject to the constraints $x \leqslant 3, y \leqslant 2, x \geqslant 0, y \geqslant 0$.



The shaded region as shown in the figure as OABC is bounded and the coordinates of corner points are (0, 0), (3, 0), (3, 2), and (0, 2), respectively.

Corner Points	**Corresponding value of Z**
(0, 0)	0
(3, 0)	33
(3, 2)	47 (Maximum)
(0, 2)	14

Hence, Z is maximise at (3, 2) and its maximum value is 47.

OR

First, we will convert the given inequations into equations, we obtain the following equations:

4x + y = 20, 2x + 3y = 30,x = 0 and y = 0

Region represented by $4x + y \geq 20$:

The line 4x + y = 20 meets the coordinate axes at A(5,0) and B(0,20) respectively. By joining these points we obtain the line 4x + y = 20

Clearly (0,0) does not satisfies the inequation $4x + y \geq 20$.

So, the region in xy plane which does not contain the origin represents the solution set of the inequation $4x + y \geq 20$

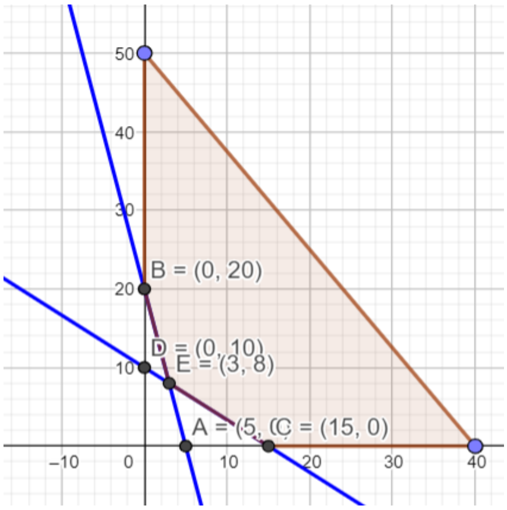
Region represented by $2x + 3y \geq 30$:

The line 2x + 3y=30 meets the coordinate axes at C(15,0) and D(0,10) respectively. By joining these points we obtain the line 2x + 3y=30 .

Clearly (0,0) does not satisfies the inequation $2x + 3y \geq 30$. So, the origin does not contain represents the solution set of the inequation $2x + 3y \geq 30$.

Region represented by $x \geq 0$ and $y \geq 0$: graph will be in first quadrant

since every point in the first quadrant satisfies these inequations. So, the first quadrant is the region represented by the in equations $x \geq 0$, and $y \geq 0$ The feasible region determined by the system of constraints, $4x + y \geq 20, 2x + 3y \geq 30, x \geq 0$, and $y \geq 0$, are as follows.



The corner points of the feasible region are B(0,20), C(15,0), E(3,8) and C(15,0)

The values of Z at these corner points are as follows.

The value of objective function at the corner point : $Z = 18x + 10y$

B(0, 20) : $18 \times 0 + 10 \times 20 = 200$

E(3, 8) : $18 \times 3 + 10 \times 8 = 134$

C(15, 0) : $18 \times 15 + 10 \times 0 = 270$

Therefore, the minimum value of Z is 134 at the point E(3,8) . Hence, x = 3 and y = 8 is the optimal solution of the given LPP.

Thus, the optimal value of objective function Z is 134.

31. According to the question,we have to prove that $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$ if $x\sqrt{1+y} + y\sqrt{1+x} = 0$

where x≠y.

we shall first write y in terms of x explicitly i.e y=f(x)

Clearly, $x\sqrt{1+y} = -y\sqrt{1+x}$

Squaring both sides,we get,

$x^2(1 + y) = y^2 (1 + x)$

$\Rightarrow x^2 + x^2y = y^2(1 + x)$

$\Rightarrow x^2 - y^2 = y^2x - x^2y$

⇒(x - y)(x + y) = -xy(x - y)

⇒(x - y)(x + y) + xy(x - y) = 0

⇒(x - y)(x + y + xy) = 0

∴Either, x - y = 0 or x + y + xy = 0

Now, x - y = 0 ⇒ x = y

But, it is given that x ≠ y

So, it is a contradiction

Therefore, x - y = 0 is rejected.

Now, consider y + xy + x = 0

⇒ y(1 + x) = -x ⇒y = $\frac{-x}{1+x}$ ....(i)

Therefore,on differentiating both sides w.r.t x, we get,

$\frac{dy}{dx} = \frac{(1+x)\times\frac{d}{dx}(-x)-(-x)\times\frac{d}{dx}(1+x)}{(1+x)^2}$ [By using quotient rule of derivative]

$\Rightarrow \frac{dy}{dx} = \frac{(1+x)(-1)+x(1)}{(1+x)^2}$

$\Rightarrow \frac{dy}{dx} = \frac{-1-x+x}{(1+x)^2}$

$\therefore \frac{dy}{dx} = \frac{-1}{(1+x)^2}$

Section D

32. The given equations are :

$y^2 = 16ax$ ...(1)

y = 4mx ....(2)

Equation (1) represent a parabola having centre at the origin and vertex along positive x−axis.

Equation (2) represents a straight line passing through the origin and making an angle of 45 with x−axis.

POINTS OF INTERSECTION :

Put y = 4mx in (1), we get

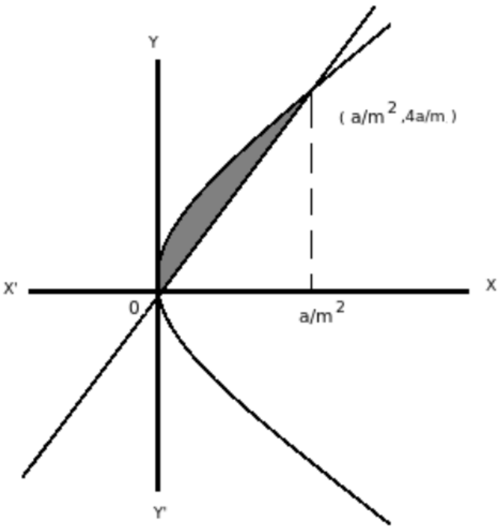
$16m^2x^2 - 16ax = 0$

$\Rightarrow 16x\,[m^2x - a] = 0$

$\Rightarrow x = 0;\ x = \frac{a}{m^2}$

When x = 0; y = 0

When $x = \frac{a}{m^2}$, then $y = \frac{4a}{m}$



Required area =Area under parabola - Area under line

$= 4\sqrt{a}\int_0^{a/m^2}\sqrt{x}dx - 4m\int_0^{a/m^2}xdx$

$= 4\sqrt{a} \times \frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^{\frac{a}{m^2}} - \frac{4m}{2}\left[x^2\right]_0^{\frac{a}{m^2}}$

$= \frac{8}{3}\frac{a^2}{m^3} - \frac{2a^2}{m^3}$

$= \frac{8}{3}\frac{a^2}{m^3} - \frac{2a^2}{m^3} = \frac{2}{3}\frac{a^2}{m^3}$

Now, area = $\frac{a^2}{12}$

So, $\frac{2}{3}\frac{a^2}{m^3} = \frac{a^2}{12}$

$\Rightarrow m^3 = 8$

$\Rightarrow m = 2$

33. Given that A = {1, 2, 3, 4},

a. Let R_1 = {(1,1), (1,2), (2,3), (2,2), (1,3), (3,3)}

R_1 is reflexive, since, (1,1) (2,2) (3,3) lie is R_1

Now, $(1,2) \in R_1$ $(2,3) \in R_1 \Rightarrow (1, 3) \in R_1$

Hence, R_1 is also transitive but $(1,2), \in R_1 \Rightarrow (2,1) \notin R_1$

So, it is not symmetric.

b. Let R_2 = {(1,2),(2,1)}. Here,$1,2,3 \in \{1,2,3\}$ but (1,1),(2,2),(3,3) are not in R.

Therefore, R is not reflexive.Now, $(1,2) \in R_2$, $(2,1) \in R_2$

So, it is symmetric.

Now $(1,2) \in R$ $(2,1) \in R$, but $(1,1) \notin R$,

therefore, R is not transitive.

c. Let R3 = {(1,2), (2,1),(1,1)(2,2),(3,3),(1,3),(3,1),(2,3)}

Clearly, R_3 is reflexive, symmetric and transitive.

OR

We have, $A = \{x \in Z : 0 \leq x \leq 12\}$ be a set and

R = {(a, b) : a = b} be a relation on A

Now,

Reflexivity: Let $a \in A$

$\Rightarrow a = a$

$\Rightarrow$ (a, a) $\in$ R

$\Rightarrow$ R is reflexive

Symmetric: Let a, b, $\in$ A and (a, b) $\in$ R

$\Rightarrow$ a = b

$\Rightarrow$ b = a

$\Rightarrow$ (b, a) $\in$ R

$\Rightarrow$ R is symmetric

Transitive: Let a, b & c $\in$ A

and let (a, b) $\in$ R and (b, c) $\in$ R

$\Rightarrow$ a = b and b = c

$\Rightarrow$ a = c

$\Rightarrow$ (a, c) $\in$ R

$\Rightarrow$ R is transitive

Since R is being reflexive, symmetric and transitive, so R is an equivalence relation.

Also we need to find the set of all elements related to 1.

Since the relation is given by, R = {(a, b): a = b}, and 1 is an element of A.

R = {(1, 1): 1 = 1}

Thus, the set of all elements related to 1 is {1}.

34. For the given system of equations, we have

D = $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

$\Rightarrow$ D = $\begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}$ [Applying $C_2 \to C_2$ - C_1 and $C_3 \to C_3$ - C_1]

$\Rightarrow$ D = (b - a) (c - a) $\begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & b+a & c+a \end{vmatrix}$

$\Rightarrow$ D = (b - a) (c - a) $\begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix}$

$\Rightarrow$ D = (b - a) (c - a) (c + a - b - a) = (b - c) (c - a) (a - b)

D_1 = $\begin{vmatrix} 1 & 1 & 1 \\ k & b & c \\ k^2 & b^2 & c^2 \end{vmatrix}$ = (b - c) (c - k) (k - b)

D_2 = $\begin{vmatrix} 1 & 1 & 1 \\ a & k & c \\ a^2 & k^2 & c^2 \end{vmatrix}$ = (k - c) (c - a) (a - k)

and, D_3 = $\begin{vmatrix} 1 & 1 & 1 \\ a & b & k \\ a^2 & b^2 & k^2 \end{vmatrix}$ = (a - b) (b - k) (k - a)

$\therefore$ x = $\frac{D_1}{D}$, y = $\frac{D_2}{D}$ and z = $\frac{D_3}{D}$

$\Rightarrow$ x = $\frac{(b-c)(c-k)(k-b)}{(b-c)(c-a)(a-b)}$, y = $\frac{(k-c)(c-a)(a-k)}{(b-c)(c-a)(a-b)}$ and z = $\frac{(a-b)(b-k)(k-a)}{(a-b)(b-c)(c-a)}$

Hence, x = $\frac{(c-k)(k-b)}{(c-a)(a-b)}$, y = $\frac{(k-c)(a-k)}{(b-c)(a-b)}$ and z = $\frac{(b-k)(k-a)}{(b-c)(c-a)}$

is the solution of given system of equations.

35. Let

$\vec{a} = l_1\hat{i} + m_1\hat{j} + n_1\hat{k}$

$\vec{b} = l_2\hat{i} + m_2\hat{j} + n_2\hat{k}$

$\vec{c} = l_3\hat{i} + m_3\hat{j} + n_3\hat{k}$

$\vec{d} = (l_1 + l_2 + l_3)\hat{i} + (m_1 + m_2 + m_3)\hat{j} + (n_1 + n_2 + n_3)\hat{k}$

Also, let α, β and γ are the angles between $\vec{a}$ and $\vec{d}$, $\vec{b}$ and $\vec{d}$, $\vec{c}$ and $\vec{d}$.

$\therefore \cos\alpha = l_1(l_1 + l_2 + l_3) + m_1(m_1 + m_2 + m_3) + n_1(n_1 + n_2 + n_3)$

$= l_1^2 + l_1l_2 + l_1l_3 + m_1^2 + m_1m_2 + m_1m_3 + n_1^3 + n_1n_2 + n_1n_3$

$= \left(l_1^2 + m_1^2 + n_1^2\right) + (l_1 l_2 + l_1 l_3 + m_1 m_2 + m_1 m_3 + n_1 n_2 + n_1 n_3)$

= 1 + 0 = 1

$[\because l_1^2 + m_1^2 + n_1^2 = 1 \text{ and } l_1 \perp l_2, l_1 \perp l_3, m_1 \perp m_2, m_1 \perp m_3, n_1 \perp n_2, n_1 \perp n_3]$

Similarly, $\cos\beta = l_2(l_1 + l_2 + l_3) + m_2(m_1 + m_2 + m_3) + n_2(n_1 + n_2 + n_3)$

= 1 + 0=1 and $\cos\gamma = 1 + 0$ =0

$\Rightarrow \cos\alpha = \cos\beta = \cos\gamma$

$\Rightarrow \alpha = \beta = \gamma$

So, the line whose direction cosines are proportional to $l_1 + l_2 + l_2$, $m_1 + m_2 + m_3$, $n_1 + n_2 + n_3$ make equal to angles with the three mutually perpendicular lines whose direction cosines are l_1, m_1, n_1, l_2, m_2, n_2 and l_3, m_3, n_3 respectively.

OR

$\vec{a}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{b}_1 = \hat{i} - 3\hat{j} + 2\hat{k}$

$\vec{a}_2 = 4\hat{i} + 5\hat{j} + 6\hat{k}$

$\vec{b}_2 = 2\hat{i} + 3\hat{j} + \hat{k}$

$\vec{a}_2 - \vec{a}_1 = 3\hat{i} + 3\hat{j} + 3\hat{k}$

$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix}$

$= -9\hat{i} + 3\hat{j} + 9\hat{k}$

$\therefore |\vec{b}_1 \times \vec{b}_2| = \sqrt{(-9)^2 + 3^2 + 9^2}$

$= \sqrt{3^2}\sqrt{3^2 + 1 + 3^2} = 3\sqrt{19}$

Required shortest distance

$= \left|\frac{(a_2 - a_1).(b_2 - b_1)}{|b_1 \times b_2|}\right| = \left|\frac{-9\times3+3\times3+9\times3}{3\sqrt{19}}\right| = \left|\frac{9}{3\sqrt{19}}\right| = \frac{3}{\sqrt{19}} units$

Section E

36. i. Let E denote the event that the student has failed in Economics and M denote the event that the student has failed in Mathematics.

$\therefore P(E) = \frac{50}{100} = \frac{1}{2}$, P(M) = $\frac{35}{100} = \frac{7}{20}$ and P(E ∩ M) = $\frac{25}{100} = \frac{1}{4}$

The probability that the selected student has failed in Economics if it is known that he has failed in Mathematics.

Required probability = $P(\frac{E}{M})$

= $\frac{P(E\cap M)}{P(M)} = \frac{\frac{1}{4}}{\frac{7}{20}}$ = $\frac{1}{4} \times \frac{20}{7} = \frac{5}{7}$

ii. Let E denote the event that student has failed in Economics and M denote the event that student has failed in Mathematics.

$\therefore P(E) = \frac{50}{100} = \frac{1}{2}$, P(M) = $\frac{35}{100} = \frac{7}{20}$ and P(E ∩ M) = $\frac{25}{100} = \frac{1}{4}$

The probability that the selected student has failed in Mathematics if it is known that he has failed in Economics.

Required probability = P(M/E)

$= \frac{P(M\cap E)}{P(E)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$

iii. Let E denote the event that the student has failed in Economics and M denote the event that the student has failed in Mathematics.

$\therefore P(E) = \frac{50}{100} = \frac{1}{2}$, P(M) = $\frac{35}{100} = \frac{7}{20}$ and P(E ∩ M) = $\frac{25}{100} = \frac{1}{4}$

The probability that the selected student has passed in Mathematics if it is known that he has failed in Economics

Required probability = P(M'/E)

$\Rightarrow P(M'/E) = \frac{P(M'\cap E)}{P(E)}$

= $\frac{P(E) - P(E\cap M)}{P(E)}$

= $\frac{\frac{1}{2} - \frac{1}{4}}{\frac{1}{2}}$

$\Rightarrow P(M'/E) = \frac{1}{2}$

OR

Let E denote the event that the student has failed in Economics and M denote the event that the student has failed in Mathematics.

$\therefore P(E)=\frac{50}{100}=\frac{1}{2}$, P(M) = $\frac{35}{100}=\frac{7}{20}$ and P(E ∩ M) = $\frac{25}{100}=\frac{1}{4}$

The probability that the selected student has passed in Economics if it is known that he has failed in Mathematics

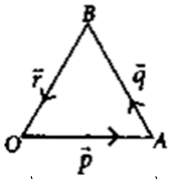
Required probability = P(E'/M)

$\Rightarrow P\left(E'/M\right)=\frac{P(E'\cap M)}{P(M)}$

= $\frac{P(M)-P(E\cap M)}{P(M)}$

= $\frac{\frac{7}{20}-\frac{1}{4}}{\frac{7}{20}} \Rightarrow P\left(E'/M\right)=\frac{2}{7}$

37. i. Let OAB be a triangle such that



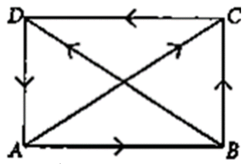
$\overrightarrow{AO}=-\vec{p}, \overrightarrow{AB}=\vec{q}, \overrightarrow{BO}=\vec{r}$

Now, $\vec{q}+\vec{r}=\overrightarrow{AB}+\overrightarrow{BO}$

$=\overrightarrow{AO}=-\vec{p}$

ii. From triangle law of vector addition,

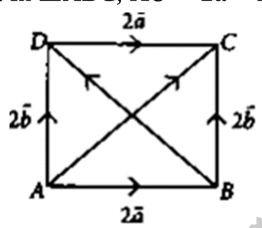
$\overrightarrow{AC}+\overrightarrow{BD}=\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{BC}+\overrightarrow{CD}$



= $\overrightarrow{AB}+2\overrightarrow{BC}+\overrightarrow{CD}$

= $\overrightarrow{AB}+2\overrightarrow{BC}-\overrightarrow{AB}=2\overrightarrow{BC}$ $[\because \overrightarrow{AB}=-\overrightarrow{CD}]$

iii. In $\triangle$ABC, $\overrightarrow{AC}$ = $2\vec{a}$ + $2\vec{b}$ ...(i)



and in $\triangle$ABD, $2\vec{b}$ = $2\vec{a}$ + $\overrightarrow{BD}$ ...(ii) [By triangle law of addition]

Adding (i) and (ii), we have $\overrightarrow{AC}+2\vec{b}=4\vec{a}+\overrightarrow{BD}+2\vec{b}$

$\Rightarrow \overrightarrow{AC}-\overrightarrow{BD}=4\vec{a}$

OR

Since T is the mid point of YZ

So, $\overrightarrow{YT}=\overrightarrow{TZ}$

Now, $\overrightarrow{XY}+\overrightarrow{XZ}=\left(\overrightarrow{XT}+\overrightarrow{TY}\right)+\left(\overrightarrow{XT}+\overrightarrow{TZ}\right)$ [By triangle law]

= $2\overrightarrow{XT}+\overrightarrow{TY}+\overrightarrow{TZ}=2\overrightarrow{XT}$ $[\because \overrightarrow{TY}=-\overrightarrow{YT}$]

38. i. We have, I(x) = $\frac{1000}{x^2}+\frac{125}{(600-x)^2}$

$\Rightarrow$ I'(x) = $\frac{-2000}{x^3}+\frac{250}{(600-x)^3}$ and

$\Rightarrow$ I''(x) = $\frac{6000}{x^4}+\frac{750}{(600-x)^4}$

For maxima/minima, I'(x) - 0

$\Rightarrow \frac{2000}{x^3}=\frac{250}{(600-x)^3}$ $\Rightarrow$ 8(600 - x)^3 = x^3

Taking cube root on both sides, we get

2(600 - x) = x $\Rightarrow$ 1200 = 3x $\Rightarrow$ x = 400

Thus, I(x) is minimum when you are at 400 feet from the strong intensity lamp post.

ii. At a distance of 200 feet from the weaker lamp post.

Since I(x) is minimum when x = 400 feet, therefore the darkest spot between the two light is at a distance of 400 feet from a stronger lamp post, i.e., at a distance of 600 - 400 = 200 feet from the weaker lamp post.

iii. $\frac{1000}{x^2} + \frac{125}{(600-x)^2}$

Since, the distance is x feet from the stronger light, therefore the distance from the weaker light will be 600 - x.

So, the combined light intensity from both lamp posts is given by $\frac{1000}{x^2} + \frac{125}{(600-x)^2}$.

OR

We know that l(x) = $\frac{1000}{x^2} + \frac{125}{(600-x)^2}$

When x = 400

l(x) = $\frac{1000}{160000} + \frac{125}{(600-400)^2}$

= $\frac{1}{160} + \frac{125}{40000} = \frac{1}{160} + \frac{1}{320} = \frac{3}{320}$ units