Solution

MATHEMATICS

Class 12 - Maths & Stats (Gen)

Section A

1. Select and write the correct answer for the following multiple-choice type of questions :

(i) **(a)** $(\sim p \wedge q) \vee \sim r$

Explanation: {

$\sim [(p \vee \sim q) \wedge r]$

$\equiv \sim (p \vee \sim q) \vee \sim r \ldots$ [De Morgan's law]

$\equiv (\sim p \wedge q) \vee \sim r \ldots$ [De Morgan's law]

(ii) **(d)** $\frac{\pi}{2}$

Explanation: {

$a = 2, b = 3$ and $\sin A = \frac{2}{3}$

Now by sine Rule, we have

$\frac{a}{\sin A} = \frac{b}{\sin B}$

$\therefore \frac{2}{\frac{2}{3}} = \frac{3}{\sin B}$

$\therefore 3 = \frac{3}{\sin B}$

$\therefore \sin B = 1$

$\therefore \angle B = \frac{\pi}{2}$

(iii) **(b)** $fg = ch$

Explanation: {

Now we know that if the combined equation $ax^2 + 2\,h'xy + by^2 + 2\,g'x + 2f'y + c = 0$ represents a pair of straight lines, then we have

$$\begin{vmatrix} a & h' & g' \\ h' & b & f' \\ g' & f' & c \end{vmatrix}$$

Now comparing the given equation $hxy + gx + fy + c = 0$ with the standard form

$ax^2 + 2\,h'xy + b^2 + 2\,g'x + 2f'y + c = 0$ we get $a = 0,\ b = 0$

$2\,h' = h \Rightarrow h' = \frac{h}{2}$

$2\,g' = g \Rightarrow g' = \frac{g}{2}$

$2f' = f \Rightarrow f' = \frac{f}{2}$

Substituting in the determinant above, we get

$$\begin{vmatrix} 0 & \frac{h}{2} & \frac{g}{2} \\ \frac{h}{2} & 0 & \frac{f}{2} \\ \frac{g}{2} & \frac{f}{2} & c \end{vmatrix} = 0$$

$\Rightarrow 0\left(0 - \frac{f^2}{4}\right) - \frac{h}{2}\left(\frac{hc}{2} - \frac{gf}{4}\right) + \frac{g}{2}\left(\frac{hf}{4} - 0\right) = 0$

$\Rightarrow \frac{-h^2c}{4} + \frac{hgf}{8} + \frac{hgf}{8} = 0$

$\Rightarrow \frac{-2h^2c + hgf + hgf}{8} = 0$

$\Rightarrow -2h^2c + 2hgf = 0$

$\Rightarrow 2\,h^2c = 2hgf$

$\Rightarrow hc = gf$

$\Rightarrow fg = ch$

(iv) **(c)** $c \wedge (p \wedge q)$

Explanation: {

$c \wedge (p \wedge q)$

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(v) **(b)** $\frac{1}{2}$

Explanation: {

$f(x) = x^5 + 2x - 3$

$\therefore f'(x) = 5x^4 + 2$

$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)}$

$f(x) = -3$ corresponds to $x = 0$.

$\therefore \left(f^{-1}\right)'(-3) = \frac{1}{f'(0)} = \frac{1}{5(0)^4+2} = \frac{1}{2}$

(vi) **(b)** $\frac{1}{12}\sin 3x + \frac{3}{4}\sin x + c$

Explanation: {

$\int \cos^3 x\ dx = \int \frac{\cos 3x + 3\cos x}{4}\ dx\ \ \ldots \left[\because \cos 3\ A = 4\cos^3\ A - 3\cos\ A\right]$

$= \frac{1}{4}\left(\frac{\sin 3x}{3} + 3\sin x\right) + c$

$= \frac{\sin 3x}{12} + \frac{3}{4}\sin x + c$

(vii) **(b)** $y\sec x = \tan x + c$

Explanation: {

$\frac{dy}{dx} = \sec x - y\tan x$

$\therefore \frac{dy}{dx} + y\tan x = \sec x$

Here, $P = \tan x$ and $Q = \sec x$

$\therefore$ I.F. $= e^{\int \tan x dx}$

$= e^{log(\sec x|}$

$= \sec x$

$\therefore$ Solution of the given equation is

$y\sec x = \int \sec x \cdot \sec x + c$

$\therefore y\sec x = \int \sec^2 x + c$

$\therefore y\sec x = \tan x + c$

(viii) **(d)** $\frac{20}{9}$

Explanation: {

$\frac{20}{9}$

2. Answer the following questions :

(i) Here we have the joint equation as

$3x^2 - 10xy - 8y^2 = 0$

$3x^2 - 12xy + 2xy - 8y^2 = 0$

$\therefore 3x(x - 4y) + 2y(x - 4y) = 0$

$\therefore (3x + 2y)(x - 4y) = 0$

$\therefore 3x + 2y = 0$ or $x - 4y = 0$

Hence, the separate equations of lines are $3x + 2y = 0$ or $x - 4y = 0$

(ii) $5\bar{a} - 3\bar{b} - 2\bar{c} = \bar{0}$

$\therefore 2\bar{c} = 5\bar{a} - 3\bar{b}$

$\therefore \bar{c} = \frac{5\bar{a} - 3\bar{b}}{2}$

$\therefore \bar{c} = \frac{5\bar{a} - 3\ \bar{b}}{5-3}$

This shows that the point C divides BA externally in the ratio 5:3.

(iii) Let $I = \int \frac{1}{x\log x \cdot \log(\log x)} dx$

Put $\log(\log x) = t$

Differentiating w.r.t. x, we get

$\frac{1}{\log x} \cdot \frac{1}{x}\ dx = dt$

$\therefore I = \int \frac{1}{t} dt = \log|t| + c$

$\therefore I = \log|\log(\log x)| + c$

(iv) Since the given differential equation cannot be expressed as a polynomial in differential coefficients, the degree is not defined.

Section B

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3. i. Let p : Nagpur is in Maharashtra.
q : Chennai is in Tamilnadu.
The symbolic form is $p \wedge q$.
ii. Let p : ΔABC is right angled at B.
$q : m\angle A + m\angle C = 90^\circ$
The symbolic form is $p \rightarrow q$.

4. Here,
$M_{11} = -1$ and $A_{11} = (-1)^{1+1}(-1) = -1$
$M_{12} = 4$ and $A_{12} = (-1)^{1+2}(4) = -4$
$M_{21} = 3$ and $A_{21} = (-1)^{2+1}(3) = -3$
$M_{22} = 1$ and $A_{22} = (-1)^{2+2}(1) = 1$
$\therefore$ The matrix of the co-factors is
$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ -3 & 1 \end{bmatrix}$

5. $(\dot{r}, \theta) \equiv \left(\frac{1}{2}, \frac{\pi}{3}\right)$ ...[Given]
Using $x = r\cos\theta$ and $y = r\sin\theta$, where (x, y)
are the required cartesian co-ordinates, we get
$x = \frac{1}{2}\cos\frac{\pi}{3}$ and $y = \frac{1}{2}\sin\frac{\pi}{3}$
$\therefore x = \frac{1}{2}\left(\frac{1}{2}\right)$ and $y = \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)$
$\therefore x = \frac{1}{4}$ and $y = \frac{\sqrt{3}}{4}$
$\therefore$ The required cartesian co-ordinates are $\left(\frac{1}{4}, \frac{\sqrt{3}}{4}\right)$.

6. Given that
$3\bar{a} + 5\bar{b} - 8\bar{c} = \bar{0}$
$\therefore 3\bar{a} = 8\bar{c} - 5\bar{b}$
$\therefore \bar{a} = \frac{8\bar{c} - 5\bar{b}}{3}$.
$\therefore \bar{a} = \frac{8\bar{c} - 5\bar{b}}{8 - 5}$
Now if we assume that A divides BC externally in the ratio $m : n$ then we have
$\bar{a} = \frac{m\bar{c} - n\bar{b}}{m - n}$
$\Rightarrow m = 8$ and $n = 5$
Hence, A divides line segment BC externally in the ratio $8 : 5$.

7. Let $\bar{a} = 2\hat{i} + 3\hat{j} - 4\hat{k}, \bar{b} = 5\hat{i} + 7\hat{j} + 5\hat{k}, \bar{c} = 4\hat{i} + 5\hat{j} - 2\hat{k}$
$\therefore$ Volume of a parallelopiped with coterminous edges $\bar{a}, \bar{b}$ and $\bar{c}$.
$= \left[\bar{a} \quad \bar{b} \quad \bar{c}\right]$
$= \begin{vmatrix} 2 & 3 & -4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{vmatrix}$
$= 2(-14 - 25) - 3(-10 - 20) + (-4)(25 - 28)$
$= 2(-39) - 3(-30) + (-4)(-3)$
$= -78 + 90 + 12 = 24$ cu.units

8. L.H.S.
$= (p \wedge q) \vee (\sim p \wedge q) \vee (p \wedge \sim q)$
$\equiv (p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge q)$ ...[Commutative law]
$\equiv [p \wedge (q \vee \sim q)] \vee (\sim p \wedge q)$ ...[Distributive law]
$\equiv (p \wedge T) \vee (\sim p \wedge q)$ ...[Complement law]
$\equiv p \vee (\sim p \wedge q)$ ...[Identity law]
$\equiv (p \vee \sim p) \wedge (p \vee q)$ ...[Distributive law]
$\equiv T \wedge (p \vee q)$ ...[Complement law]
$\equiv p \vee q$ ...[Identity law]
$=$ R.H.S

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9. $f(x) = x^3 + 10x + 7$

   $\therefore f'(x) = 3x^2 + 10$

   Here, $3x^2 \geq 0$ for all $x \in R$ and $10 > 0$.

   $\therefore 3x^2 + 10 > 0$

   $\therefore f'(x) > 0$ for all $x \in R$

   Thus, $f(x)$ is a strictly increasing function for all $x \in R$.

10. Let $I = \int_0^{\frac{\pi}{2}} \sin^2 x \, dx = \int_0^{\frac{\pi}{2}} \frac{1-\cos 2x}{2} \, dx$

    $= \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 - \cos 2x) dx$

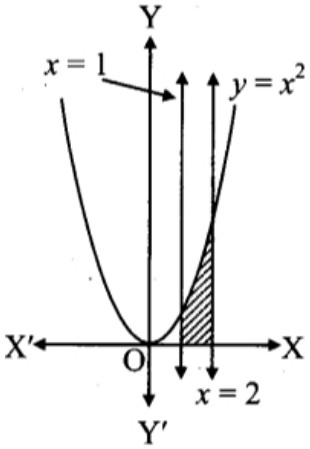
    $= \frac{1}{2} \left[x - \frac{\sin 2x}{2} \right]_0^{\frac{\pi}{2}}$

    $= \frac{1}{2} \left[\left(\frac{\pi}{2} - 0 \right) - \frac{1}{2} (\sin \pi - \sin 0) \right]$

    $= \frac{1}{2} \left[\frac{\pi}{2} - \frac{1}{2} (0 - 0) \right]$

    $\therefore I = \frac{\pi}{4}$

11. Required area



    $= \int_1^2 y dx$

    $= \int_1^2 x^2 \, dx$

    $= \frac{1}{3} \left[x^3 \right]_1^2$

    $= \frac{1}{3} \left(2^3 - 1^3 \right) = \frac{1}{3} (8 - 1)$

    $= \frac{7}{3}$ sq. units

12. $\frac{dy}{dx} + y = e^{-x}$

    The given equation is of the form

    $\frac{dy}{dx} + Py = Q$, where

    $P = 1$ and $Q = e^{-x}$

    $\therefore$ I.F. $= e^{\int P dx}$

    $= e^{\int 1 dx} = e^x$

    $\therefore$ The integrating factor (I.F.) is e^x.

13. Let $X =$ profit for each type of cake

    (according to the profit)

    $P(X = 3) = 10\% = \frac{10}{100} = 0.1$

    $P(X = 2.5) = 5\% = \frac{5}{100} = 0.05$

    $P(X = 2) = 25\% = \frac{25}{100} = 0.25$

    $P(X = 1.5) = 45\% = \frac{45}{100} = 0.45$

    $P(X = 1) = 15\% = \frac{15}{100} = 0.15$

    $\therefore$ The probability distribution table is as follows:

X	3	2.5	2	1.5	1

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$P(X)$	0.1	0.05	0.25	0.45	0.15

$E(X) = \sum x_i \cdot P(x_i)$

$= 3(0.1) + 2.5(0.05) + 2(0.25) + 1.5(0.45) + 1(0.15)$

$= 0.3 + 0.125 + 0.5 + 0.675 + 0.15$

$= 1.75$

$\therefore$ Expected profit per cake $= ₹1.75$

14. Let $I = \int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

$= \int_0^{\frac{\pi}{2}} \frac{1+\cos 2x}{2} \, dx$

$= \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 + \cos 2x) dx$

$= \frac{1}{2} \left[x + \frac{\sin 2x}{2} \right]_0^{\frac{\pi}{2}}$

$= \frac{1}{2} \left[\left(\frac{\pi}{2} - 0 \right) + \frac{1}{2} (\sin \pi - \sin 0) \right]$

$= \frac{1}{2} \left[\frac{\pi}{2} + \frac{1}{2} (0 - 0) \right]$

$\therefore I = \frac{\pi}{4}$

Section C

15. According to cosine Rule, we have

$\cos B = \frac{c^2+a^2-b^2}{2ca}$

and $\cos C = \frac{a^2+b^2-c^2}{2ab}$

$\therefore$ L.H.S. $= a(b \cos C - c \cos B)$

$= a \left[b \left(\frac{a^2+b^2-c^2}{2ab} \right) - c \left(\frac{c^2+a^2-b^2}{2ca} \right) \right]$

$= a \left[\frac{\left(a^2+b^2-c^2 \right)}{2a} - \frac{\left(c^2+a^2-b^2 \right)}{2a} \right]$

$= a \left[\frac{a^2+b^2-c^2-c^2-a^2+b^2}{2a} \right]$

$= \frac{2b^2-2c^2}{2}$

$= b^2 - c^2$

$=$ R.H.S.

16. $\sin 2x + \sin 4x + \sin 6x = 0$

$\therefore (\sin 2x + \sin 6x) + \sin 4x = 0$

$\therefore 2 \sin \left(\frac{6x+2x}{2} \right) \cos \left(\frac{6x-2x}{2} \right) + \sin 4x = 0$

$\therefore 2 \sin 4x \cos 2x + \sin 4x = 0$

$\therefore \sin 4x (2 \cos 2x + 1) = 0$

$\therefore \sin 4x = 0$ or $\cos 2x = -\frac{1}{2}$

$\therefore \sin 4x = 0$ or

$\cos 2x = -\cos \left(\frac{\pi}{3} \right) = \cos \left(\pi - \frac{\pi}{3} \right) = \cos \left(\frac{2\pi}{3} \right)$

Since, $\sin \theta = 0$ implies $\theta = n\pi$ and $\cos \theta = \cos \alpha$ implies $\theta = 2n\pi \pm \alpha, n \in Z$.

$\therefore 4x = n\pi$ or $2x = 2\, m\pi \pm \frac{2\pi}{3}$

$\therefore$ The required general solution is $x = \frac{n\pi}{4}$ or $x = m\pi \pm \frac{\pi}{3}$, where $n, m \in Z$.

17. The given joint equation is $ax^2 + 2\, hxy + by^2 = 0$

Let m_1 and m_2 be the slopes of the lines represented by given equation.

$\therefore m_1 + m_2 = \frac{-2h}{b}$ and $m_1 m_2 = \frac{a}{b}$ ...(i)

It is given that the required lines are perpendicular to the given lines

$\therefore$ Their slopes are $-\frac{1}{m_1}$ and $-\frac{1}{m_2}$

Required lines pass through the origin, therefore their equations are

$y = -\frac{1}{m_1} x$ and $y = -\frac{1}{m_2} x$

$\therefore x + m_1 y = 0$ and $x + m_2 y = 0$

$\therefore$ Their combined equation is

$(x + m_1 y)(x + m_2 y) = 0$

$\therefore x^2 + m_2 xy + m_1 xy + m_1\, m_2 y^2 = 0$

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$\therefore x^2 + (m_1 + m_2)\,xy + m_1\, m_2 y^2 = 0$

$\therefore x^2 + \left(\frac{-2\,h}{b}\right) xy + \left(\frac{a}{b}\right) y^2 = 0 \ldots [\ \text{from (i)}\ ]$

$\therefore bx^2 - 2\,hxy + ay^2 = 0$

Hence, the required joint equation of the pair of lines perpendicular to $ax^2 + 2hxy + by^2 = 0$ is $bx^2 - 2\,hxy + ay^2 = 0$.

18. Here we have the equation of the lines in cartesian form as $\frac{x-1}{4} = \frac{y-3}{1} = \frac{z}{8}$ and $\frac{x-2}{2} = \frac{y+1}{2} = \frac{z-4}{1}$

$\therefore$ The lines are parallel to the vectors $\overline{b_1} = 4\hat{i} + \hat{j} + 8\hat{k}$ and $\overline{b_2} = 2\hat{i} + 2\hat{j} + \hat{k}$

Let 'θ' be the angle between the lines

$\therefore$ we have

$\cos\theta = \frac{\overline{b_1} \cdot \overline{b_2}}{|\overline{b_1}||\overline{b_2}|}$

$\therefore \cos\theta = \frac{4.2+1.2+8.1}{\sqrt{4^2+1^2+8^2}\sqrt{2^2+2^2+1^2}}$

$= \frac{8+2+8}{\sqrt{16+1+64}\sqrt{4+4+1}}$

$= \frac{18}{\sqrt{81}\sqrt{9}} = \frac{18}{9.3}$

$\therefore \cos\theta = \frac{2}{3}$

$\therefore \theta = \cos^{-1}\left(\frac{2}{3}\right)$

19. Let L_1 and L_2 be the two lines with direction ratios $-2, 1, -1$ and $-3, -4, 1$ respectively.

Let the direction ratios of the vector perpendicular to L_1 and L_2 be a, b, c.

$\therefore -2a + b - c = 0$

and $-3a - 4b + c = 0$

$\frac{a}{\begin{vmatrix} 1 & -1 \\ -4 & 1 \end{vmatrix}} = \frac{-b}{\begin{vmatrix} -2 & -1 \\ -3 & 1 \end{vmatrix}} = \frac{c}{\begin{vmatrix} -2 & 1 \\ -3 & -4 \end{vmatrix}}$

$\therefore \frac{a}{1-4} = \frac{-b}{-2-3} = \frac{c}{8+3}$

$\therefore \frac{a}{-3} = \frac{b}{5} = \frac{c}{11}$

$\therefore$ Direction ratios of the required vector are $-3, 5, 11$.

$\therefore$ The direction cosines of the line are

$l = \frac{-3}{\sqrt{(-3)^2+5^2+11^2}} = \frac{-3}{\sqrt{9+25+121}} = \frac{-3}{\sqrt{155}}$

$m = \frac{5}{\sqrt{(-3)^2+5^2+11^2}} = \frac{5}{\sqrt{9+25+121}} = \frac{5}{\sqrt{155}}$

$n = \frac{11}{\sqrt{(-3)^2+5^2+11^2}} = \frac{11}{\sqrt{9+25+121}} = \frac{11}{\sqrt{155}}$

$\therefore$ The direction cosines of the line are $\frac{-3}{\sqrt{155}}, \frac{5}{\sqrt{155}}, \frac{11}{\sqrt{155}}$.

20. Here we have the points as $A(3, 4, -7)$ and $B(6, -1, 1)$

$\therefore$ The cartesian equation of the line passing through A and B is

$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$

$\therefore \frac{x-3}{6-3} = \frac{y-4}{-1-4} = \frac{z+7}{1-(-7)}$

$\therefore \frac{x-3}{3} = \frac{y-4}{-5} = \frac{z+7}{8}$

for vector form of the equation:

Let $\frac{x-3}{3} = \frac{y-4}{-5} = \frac{z+7}{8} = \lambda$

$\therefore x = 3\lambda + 3, y = -5\lambda + 4, z = 8\lambda - 7$

Now the equation in vector form is

$\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\overline{r} = (3\lambda + 3)\hat{i} + (-5\lambda + 4)\hat{j} + (8\lambda - 7)\hat{k}$

$= (3\hat{i} + 4\hat{j} - 7\hat{k}) + \lambda(3\hat{i} - 5\hat{j} + 8\hat{k})$

$\therefore \overline{r} = (3\hat{i} + 4\hat{j} - 7\hat{k}) + \lambda(3\hat{i} - 5\hat{j} + 8\hat{k})$

21. $y = \tan^{-1}\left(\frac{8x}{1-15x^2}\right)$

$= \tan^{-1}\left(\frac{5x+3x}{1-(5x)(3x)}\right)$

$= \tan^{-1} 5x + \tan^{-1} 3x \ \ldots \left[\because \tan^{-1}\left(\frac{a+b}{1-ab}\right) = \tan^{-1} a + \tan^{-1} b\right]$

Differentiating w. r. t. x, we get

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$\frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1} 5x + \tan^{-1} 3x\right)$

$= \frac{1}{1+(5x)^2} \cdot \frac{d}{dx}(5x) + \frac{1}{1+(3x)^2} \cdot \frac{d}{dx}(3x)$

$= \frac{1}{1+25x^2} \cdot (5) + \frac{1}{1+9x^2} \cdot 3$

$\therefore \frac{dy}{dx} = \frac{5}{1+25x^2} + \frac{3}{1+9x^2}$

22. Let $I = \int \frac{x+1}{(x+2)(x+3)} dx$

Let $\frac{x+1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$

$\therefore x + 1 = A(x + 3) + B(x + 2)$ ...(i)

Putting $x = -3$ in (i), we get

$-3 + 1 = B(-3 + 2)$

$\therefore B = 2$

Putting $x = -2$ in (i), we get

$-2 + 1 = A(-2 + 3)$

$\therefore A = -1$

$\therefore \frac{x+1}{(x+2)(x+3)} = \frac{-1}{x+2} + \frac{2}{x+3}$

$\therefore I = \int \left(\frac{-1}{x+2} + \frac{2}{x+3}\right) dx = \int \left(\frac{2}{x+3} - \frac{1}{x+2}\right) dx$

$= 2\log|x + 3| - \log|x + 2| + c$

$= \log\left|(x + 3)^2\right| - \log|x + 2| + c$

$\therefore I = \log\left|\frac{(x+3)^2}{x+2}\right| + c$

23. $\frac{dy}{dx} - y = e^x$

The given equation is of the form $\frac{dy}{dx} + Py = Q$,

where $P = -1$ and $Q = e^x$

I.F. $= e^{\int P dx} = e^{\int -1 dx} = e^{-x}$

$\therefore$ Solution of the given equation is

y (I.F.) $= \int Q$ (I.F.) $dx + c$

$\therefore y \cdot e^{-x} = \int e^x \cdot e^{-x}\, dx + c$

$\therefore y.\, e^{-x} = \int 1.\, dx + c$

$\therefore ye^{-x} = x + c$ ...(i)

When $x = 0, y = 1$, we have

$1.\, e^0 = 0 + c$

$\therefore c = 1$

Substituting $c = 1$ in (i), we get

$ye^{-x} = x + 1$

$\therefore y = (x + 1)e^x$, which is the required particular solution.

24. $f(x) = 3x + \frac{1}{3x}$

$\therefore f'(x) = 3 - \frac{1}{3x^2}$

$f(x)$ is decreasing, if $f'(x) \leq 0$

$\therefore 3 - \frac{1}{3x^2} \leq 0$

$\therefore \frac{9x^2-1}{3x^2} \leq 0$

$\therefore 9x^2 - 1 \leq 0$

$\therefore (3x + 1)(3x - 1) \leq 0$

$\therefore x \in \left[-\frac{1}{3}, \frac{1}{3}\right] - \{0\}$

Thus, $f(x)$ is decreasing for $x \in \left[-\frac{1}{3}, \frac{1}{3}\right] - \{0\}$.

25. Let X be the number of heads out of 8 tosses.

P(getting head) $= p = \frac{1}{2}, q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

Given $n = 8$

$\therefore X \sim B(n, p) \equiv X \sim B\left(8, \frac{1}{2}\right)$

The p.m.f. of X is given by

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$P(X=x)=p(x)={}^{8}C_{x}\left(\frac{1}{2}\right)^{x}\left(\frac{1}{2}\right)^{8-x}$,

$x=0,1,2,\ldots,8$

i. $P($ no head $)=P(X=0)$

$\therefore P(X=0)={}^{8}C_{0}\left(\frac{1}{2}\right)^{0}\left(\frac{1}{2}\right)^{8}$

$=\left(\frac{1}{2}\right)^{8}$

$=\frac{1}{256}$

$=0.0039$

ii. $P($ at least one head $)=P(X\geq 1)$

$\therefore P(X\geq 1)=1-P(\text{no head})$

$=1-P(X<1)$

$=1-P(X=0)$

$=1-{}^{8}C_{0}\left(\frac{1}{2}\right)^{0}\left(\frac{1}{2}\right)^{8}$

$=1-(1)(1)\times\frac{1}{256}$

$=\frac{255}{256}$

$=0.9961$

26. $P(X<1)=\int_{-3}^{1}f(x)dx$

$=\int_{-3}^{1}\frac{x^2}{18}\,dx$

$=\frac{1}{18}\left[\frac{x^3}{3}\right]_{-3}^{1}$

$=\frac{1}{18}\left(\frac{1}{3}+\frac{27}{3}\right)$

$=\frac{28}{54}$

$=\frac{14}{27}$

$\therefore P(X<1)=0.5185$

$P(|X|<1)=P(-1<X<1)$

$=\int_{-1}^{1}f(x)dx$

$=\int_{-1}^{1}\frac{x^2}{18}\,dx$

$=\frac{1}{18}\left[\frac{x^3}{3}\right]_{-1}^{1}$

$=\frac{1}{18}\left(\frac{1}{3}+\frac{1}{3}\right)$

$\therefore P(|X|<1)=\frac{1}{27}=0.03703$

Section D

27. To draw the feasible region, construct table as follows:

Inequality	$2x+y\geq 7$	$2x+3y\geq 15$	$y\geq 2$
Corresponding equation (of line)	$2x+y=7$	$2x+3y=15$	$y=2$
Intersection of line with X-axis	$\left(\frac{7}{2},0\right)$	$\left(\frac{15}{2},0\right)$	-
Intersection of line with Y-axis	$(0,7)$	$(0,5)$	$(0,2)$
Region	Non-origin side	Non-origin side	Non-origin side

Shaded portion EABCY is the feasible region, whose vertices are A, B and $C(0,7)$

A is the point of intersection of the lines $y=2$ and $2x+3y=15$

Putting $y=2$ in $2x+3y=15$, we get

$2x+6=15$

$\therefore x=\frac{9}{2}$

$\therefore A\equiv\left(\frac{9}{2},2\right)$

B is the point of intersection of the lines $2x+y=7$ and $2x+3y=15$.

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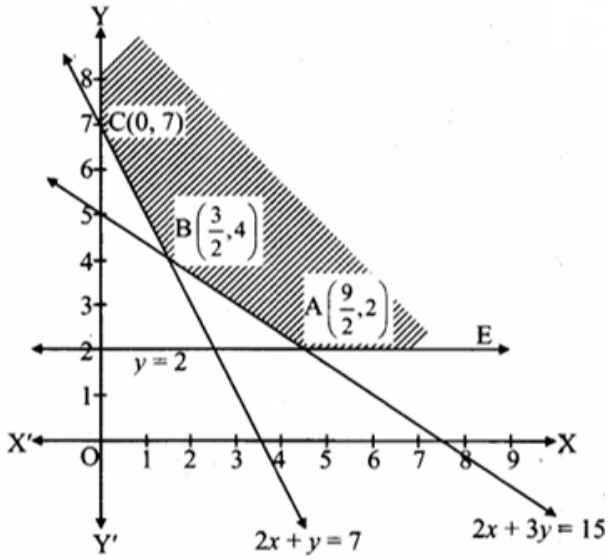

Solving the above equations, we get

$x = \frac{3}{2}, y = 4$

$\therefore B \equiv \left(\frac{3}{2}, 4\right) \equiv (1.5, 4)$

Here, the objective function is $Z = 8x + 10y$,

Z at $A\left(\frac{9}{2}, 2\right) = 8\left(\frac{9}{2}\right) + 10(2)$

$= 36 + 20 = 56$

Z at $B\left(\frac{3}{2}, 4\right) = 8\left(\frac{3}{2}\right) + 10(4) = 12 + 40 = 52$

Z at $C(0, 7) = 8(0) + 10(7) = 70$

$\therefore Z$ has minimum value 52 at $B\left(\frac{3}{2}, 4\right)$ i.e. at $(1.5, 4)$

$\therefore Z$ is minimum, when $x = \frac{3}{2}$ i.e 1.5 and $y = 4$.

28. Let the three numbers be x, y, z.

According to the first condition,

$x + y + z = 2$

According to the second condition,

$(x + z) - 2y = 8$

i.e., $x - 2y + z = 8$

According to the third condition,

$3x + y + z = 4$

Matrix form of the given system of equations is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 4 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 3R_1$, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 0 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix}$$

Applying $R_2 \to \left(-\frac{1}{3}\right) R_2$ and $R_3 \to \left(-\frac{1}{2}\right) R_3$, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$$

Hence, the original matrix is reduced to an upper triangular matrix.

$\therefore$ By equality of matrices, we get

$x + y + z = 2$ ...(i)

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$y = -2$ ...(ii)

$z = 3$ ...(iii)

Putting $y = -2$ and $z = 3$ in equation (i), we get

$x - 2 + 3 = 2$

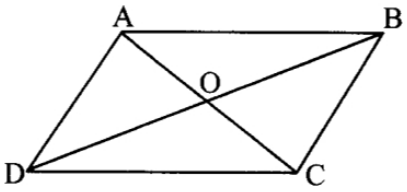
$\therefore x = 1$

Hence, 1, -2 and 3 are the required numbers.

29. Let the position vectors of points A, B, C and D be $\bar{a}, \bar{b}, \bar{c}$ and $\bar{d}$ respectively

$\therefore$ we have $\bar{a} = \hat{i} + 2\hat{j} - \hat{k}, \bar{b} = 8\hat{i} - 3\hat{j} - 4\hat{k}$, $\bar{c} = 5\hat{i} - 4\hat{j} + \hat{k}$, $\bar{d} = -2\hat{i} + \hat{j} + 4\hat{k}$

Now consider the parallelogram A, B, C, D as shown below



Let the point of intersection of diagonals AC and BD be O.

Now we know that diagonals of a parallelogram bisect each other. Therefore, we need to prove that O is the midpoint of both AC and BD.

Now midpoint of $AC = \frac{\bar{a}+\bar{c}}{2}$

$= \frac{\hat{i}+2\hat{j}-\hat{k}+5\hat{i}-4\hat{j}+\hat{k}}{2}$

$= \frac{6\hat{i}-2\hat{j}}{2}$

$= 3\hat{i} - \hat{j}$ ...(i)

Midpoint of $BD = \frac{\bar{b}+\bar{d}}{2}$

$= \frac{8\hat{i}-3\hat{j}-4\hat{k}-2\hat{i}+\hat{j}+4\hat{k}}{2}$

$= \frac{6\hat{i}-2\hat{j}}{2}$

$= 3\hat{i} - \hat{j} \ldots$ (ii)

Thus, we can see from equations (i) and (ii) that the position vector of midpoint O is $3\hat{i} - \hat{j}$. That is, it coincides in both the cases.

Hence, point O is the midpoint of diagonals AC and BD both.

$\therefore$ Diagonals AC and BD bisect each other.

$\therefore$ Hence, quadrilateral ABCD is a parallelogram.

30. Let $A \equiv (1, 1, 2)$

$\therefore \bar{a} = \hat{i} + \hat{j} + 2\hat{k}$

Equation of the given plane is $x + 2y + 2z = -8$

Let $x + 2y + 2z = r$ be the required plane. ...(i)

The d.r.s. of the normal to the plane are $1, 2, 2$.

$\therefore$ Direction cosines are

$l = \frac{1}{\sqrt{1^2+2^2+2^2}}, m = \frac{2}{\sqrt{1^2+2^2+2^2}}$,

$n = \frac{2}{\sqrt{1^2+2^2+2^2}}$

$\therefore l = \frac{1}{3}, m = \frac{2}{3}, n = \frac{2}{3}$

Cartesian equation of the required planes in a normal form is $lx + my + nz = p$

$\therefore \frac{1}{3}x + \frac{2}{3}y + \frac{2}{3}z = \frac{r}{3}$

$\therefore p = \frac{r}{3}$

Now, $\bar{a} \cdot \hat{n} = (\hat{i} + \hat{j} + 2\hat{k}) \cdot \left(\frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}\right)$

$= \frac{(\hat{i}+\hat{j}+2\hat{k})\cdot(\hat{i}+2\hat{j}+2\hat{k})}{3}$

$= \frac{(1)(1)+(1)(2)+(2)(2)}{3} = \frac{1+2+4}{3} = \frac{7}{3}$

$\therefore$ The distance of the point $(1, 1, 2)$ from the given plane is $|\widehat{a} \cdot \hat{n} - p|$

$\therefore |\bar{a} \cdot \hat{n} - p| = 2$ ...[Given]

$\therefore \left|\frac{7}{3} - \frac{r}{3}\right| = 2$

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$\therefore \frac{7-r}{3} = \pm 2$

$\therefore 7 - r = 6$ or $7 - r = -6$

$\therefore r = 1$ or $r = 13$

Putting values of r in equation (i), we get $x + 2y + 2z = 1$ and $x + 2y + 2z = 13$, which are the required equations of the planes.

31. Proof:

x and y are differentiable functions of t.

Let there be a small increment δt in the value of t.

Correspondingly, there should be a small increments $\delta x, \delta y$ in the values of x and y respectively.

As $\delta t \to 0, \delta x \to 0, \delta y \to 0$

Consider, $\frac{\delta y}{\delta x} = \frac{\frac{\delta y}{\delta t}}{\frac{\delta x}{\delta t}}$

Taking $\lim_{\delta \to 0}$ on both sides, we get

$\lim_{\delta \to 0} \frac{\delta y}{\delta x} = \frac{\lim_{\delta i \to 0} \frac{\delta y}{\delta t}}{\lim_{\delta t \to 0} \frac{\delta x}{\delta t}}$

Since x and y are differentiable functions of t, $\lim_{\delta \to 0} \frac{\delta y}{\delta t} = \frac{dy}{dt}$ exists and is finite.

$\lim_{\delta \to 0} \frac{\delta x}{\delta t} = \frac{dx}{dt}$ exists and is finite.

$\therefore \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)$ (as $\delta t \to 0, \delta x \to 0$)

$\therefore$ Limits on right hand side exists and are finite.

$\therefore$ Limits on the left hand side should also exists and be finite.

$\therefore \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$ exists and is finite.

$\therefore \frac{dy}{dx} = \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right), \frac{dx}{dt} \neq 0$

$x = a\cos t, y = a\sin t$

$\therefore \frac{dx}{dt} = -a\sin t$ and $\frac{dy}{dt} = a\cos t$

$\therefore \frac{dy}{dx} = \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = -\frac{a\cos t}{a\sin t} = -\cot t$

$x = a\cos^2 t$ and $y = a\sin^2 t$

$\therefore \frac{dx}{dt} = \frac{d}{dt}\left(a\cos^2 t\right) = a[2\cos t(-\sin t)]$

$\therefore \frac{dx}{dt} = -2a\sin t\cos t$

and $\frac{dy}{dt} = \frac{d}{dt}\left(a\sin^2 t\right)$

$\therefore \frac{dy}{dt} = a(2\sin t\cos t) = 2a\sin t\cos t$

$\therefore \frac{dy}{dx} = \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \frac{2a\sin t\cos t}{-2a\sin cost} = -1$

32. Let the length and breadth of the rectangle be l and b metres respectively.

$\therefore$ Perimeter of rectangle $= 2(l + b) = 36$ metres

$\therefore l + b = 18$

$\therefore b = 18 - l$ ...(i)

Area of rectangle $= l \times b = l(18 - l)$

$\therefore$ Area of rectangle $= f(l) = 18l - l^2$

$\therefore f'(l) = 18 - 2l$

$\therefore f''(l) = -2$

Now, $f'(l) = 0$

$\therefore 18 - 2l = 0$

$\therefore 18 = 2l$

$\therefore l = 9$

For $l = 9$,

$f''(9) = -2 < 0$

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$\therefore f(l)$, i.e., area is maximum when $l = 9$ and $b = 18 - 9 = 9$ metres ...[From (i)]

Thus, the area of the rectangle is maximum if length $=$ breadth $= 9$ metres.

33. Let $I = \int \frac{5e^x}{(e^x+1)(e^{2x}+9)}dx$

Put $e^x = t$

$\therefore e^x dx = dt$

$\therefore I = \int \frac{5dt}{(t+1)(t^2+9)}$

Let $\frac{5}{(t+1)(t^2+9)} = \frac{A}{t+1} + \frac{Bt+C}{t^2+9}$

$\therefore 5 = A\left(t^2+9\right) + (Bt+C)(t+1)$

Putting $t = -1$ in (i), we get

$5 = A\left[(-1)^2+9\right]$

$\therefore 5 = 10\ A$

$\therefore A = \frac{1}{2}$

Putting $t = 0$ in (i), we get

$5 = A(0+9) + (0+C)(0+1)$

$\therefore 5 = 9\ A + C$

$\therefore 5 = 9\left(\frac{1}{2}\right) + C$

$\therefore C = \frac{1}{2}$

Putting $t = 1$ in (i), we get

$5 = A\left(1^2+9\right) + (B+C)(1+1)$

$\therefore 5 = 10\ A + 2\ B + 2C$

$\therefore 5 = 10\left(\frac{1}{2}\right) + 2B + 2\left(\frac{1}{2}\right)$

$\therefore -1 = 2B$

$\therefore B = -\frac{1}{2}$

$\therefore \frac{5}{(t+1)(t^2+9)} = \frac{\frac{1}{2}}{t+1} + \frac{-\frac{1}{2}t+\frac{1}{2}}{t^2+9}$

$\therefore I = \int \left(\frac{\frac{1}{2}}{t+1} + \frac{\frac{-1}{2}t+\frac{1}{2}}{t^2+9}\right)dt$

$= \frac{1}{2}\int \left(\frac{1}{t+1} + \frac{-t+1}{t^2+9}\right)dt$

$= \frac{1}{2}\left[\int \frac{1}{t+1}dt - \int \frac{t}{t^2+9}dt + \int \frac{1}{t^2+9}dt\right]$

$= \frac{1}{2}\left[\int \frac{1}{t+1}dt - \frac{1}{2}\int \frac{2t}{t^2+9}dt + \int \frac{1}{t^2+3^2}dt\right]$

$= \frac{1}{2}\left[\log|t+1| - \frac{1}{2}\log|t^2+9| + \frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right)\right] + c$

$\therefore I = \frac{1}{2}\log|e^x+1| - \frac{1}{4}\log|e^{2x}+9| + \frac{1}{6}\tan^{-1}\left(\frac{e^x}{3}\right) + c$

34. **Proof:**

Consider R.H.S.: $\int_0^a f(a-x)dx$

Let $I = \int_0^a f(a-x)dx$

Put $a - x = t$

$\therefore -dx = dt$

$\therefore dx = -dt$

When $x = 0, t = a - 0 = a$

and when $x = a, t = a - a = 0$

$\therefore I = \int_a^0 f(t)(-dt)$

$= -\int_a^0 f(t)dt$

$= \int_0^a f(t)dt$ ... $\left[\because \int_a^b f(x)dx = -\int_b^a f(x)dx\right]$

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$= \int_0^a f(x)dx$ ... $\left[\because \int_a^b f(x)dx = \int_a^b f(t)dt\right]$

$=$ L.H.S.

$\therefore \int_0^a f(x)dx = \int_0^a f(a-x)dx$

Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}\, dx$ ...(i)

$= \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)} dx \cdots \left[\because \int_0^2 \mathrm{f}(x)\mathrm{d}x = \int_0^2 \mathrm{f}(\mathrm{a}-x)\mathrm{d}x\right]$

$\therefore I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x+\sin x}\, dx$ ...(ii)

Adding (i) and (ii), we get

$2I = \int_0^{\frac{\pi}{2}} \frac{\sin x+\cos x}{\sin x+\cos x}\, dx = \int_0^{\frac{\pi}{2}} 1.\, dx = [x]_0^{\frac{\pi}{2}}$.

$\therefore 2I = \frac{\pi}{2} - 0$

$\therefore I = \frac{\pi}{4}$

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