Solution

MATHEMATICS

Class 12 - Maths & Stats (Gen)

Section A

1. Select and write the correct answer for the following multiple-choice type of questions :

(i) **(c)** $c \wedge (p \wedge q)$

Explanation: {

$c \wedge (p \wedge q)$

(ii) **(d)** $\frac{\pi}{6}, \frac{11\pi}{6}$

Explanation: {

Here we have $\sec x = \frac{2}{\sqrt{3}}$

$\therefore \sec x = \sec \frac{\pi}{6}$ and $\sec x = \sec\left(2\pi - \frac{\pi}{6}\right) = \sec \frac{11\pi}{6}$

Such that $0 < \frac{\pi}{6} < 2\pi$ and $0 < \frac{11\pi}{6} < 2\pi$

Hence, the principal solution are $\frac{\pi}{6}$ and $\frac{11\pi}{6}$

(iii) **(d)** $\frac{11}{2}$

Explanation: {

Comparing given equation with the standard form $ax^2 + 2\,hxy + by^2 = 0$, we get

$a = 3, 2\,h = k, b = 2$

$\therefore$ The auxiliary equation is

$bm^2 + 2hm + a = 0$

$\therefore 2\,m^2 + km + 3 = 0$ ...(i)

Now since one of the equations is

$2x + y = 0$

its slope is $m = -2$

Substituting $m = -2$ in equation (i), we get

$2(-2)^2 + (-2)k + 3 = 0$

$\therefore 8 - 2k + 3 = 0$

$\therefore 11 - 2k = 0$

$\therefore 2k = 11$

$\therefore k = \frac{11}{2}$

(iv) **(d)** $\forall x \in A$ such that $x + 6 \geq 9$

Explanation: {

For $x = 2 \in A$, we have $x + 6 = 8 < 9$

i.e., $x = 2$ does not satisfy the condition $x + 6 \geq 9$

(v) **(d)** $\frac{1}{t^2}$

Explanation: {

$x = a^4$

$\therefore \frac{dx}{dt} = 4at^3$

$y = 2at^2$

$\therefore \frac{dy}{dt} = 4at$

$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{4at}{4at^3} = \frac{1}{t^2}$

(vi) **(d)** $\frac{1}{3}\tan^{-1}(3x) + c$

Explanation: {

$\int \frac{dx}{9x^2+1} = \int \frac{dx}{(3x)^2+1^2}$

$= \frac{1}{3}\tan^{-1}(3x) + c$

(vii) **(a)** $x = e^{\text{ct}}$

Explanation: {

$\frac{dx}{dt} = \frac{x \log x}{t}$

$\therefore \frac{dx}{x \log x} = \frac{dt}{t}$

Integrating on both sides, we get

$\int \frac{dx}{x \log x} = \int \frac{dt}{t}$

$\therefore \log|\log x| = \log|t| + \log|c|$

$\therefore \log|\log x| = \log|ct|$

$\therefore \log x = ct$

$\therefore x = e^{ct}$

(viii) **(b)** 18

Explanation: {

$E(X) = np$ and $\text{Var}(X) = npq$

$\therefore \frac{\text{Var}(X)}{E(X)} = \frac{npq}{np}$

$\therefore \frac{4}{12} = q$

$\therefore q = \frac{1}{3}$

$\therefore p = 1 - q = 1 - \frac{1}{3} = \frac{2}{3}$

Now, $np = 12$

$\therefore n\left(\frac{2}{3}\right) = 12$

$\therefore n = 18$

2. Answer the following questions :

(i) Given equation of the lines is $kx^2 + 4xy - 4y^2 = 0$ Here, $a = k, b = -4$

Since the given lines are perpendicular, $a + b = 0$

$\therefore k + (-4) = 0$

$\therefore k = 4$

(ii) Since the given vectors are collinear.

$\therefore \frac{2}{p} = \frac{-3}{6} = \frac{4}{-8}$

$\therefore \frac{2}{p} = \frac{-3}{6}$

$\therefore p = -4$

(iii) Let $I = \int \frac{1}{\sin x \cdot \cos^2 x}\, dx$

$= \int \frac{\sin^2 x + \cos^2 x}{\sin x \cos^2 x}\, dx$

$= \int \frac{\sin x}{\cos^2 x}\, dx + \int \frac{1}{\sin x}\, dx$

$= \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}\, dx + \int \operatorname{cosec} x\, dx$

$= \int \tan x \cdot \sec x\, dx + \int \operatorname{cosec} x\, dx$

$\therefore I = \sec x + \log\left|\tan\left(\frac{x}{2}\right)\right| + c$

(iv) Since the given differential equation cannot be expressed as a polynomial in differential coefficients, the degree is not defined.

Section B

3. $(p \wedge q) \vee \sim q \equiv p \vee \sim q$

1	2	3	4	5	6
P	q	$\sim q$	p $\wedge q$	$(p \wedge q) \vee \sim q$	p $\wedge \sim$ q
T	T	F	T	T	T
T	F	T	F	T	T
F	T	F	F	F	F
F	F	T	F	T	T

In the above truth table, the entries in the columns 5 and 6 are identical.

$\therefore (p \wedge q) \vee \sim q \equiv p \vee \sim q$

4. Since, $AX = I$

$\therefore \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Applying $R_2 \to R_2 - 3R_1$,

$\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} X = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

Applying $R_2 \to \left(-\frac{1}{2}\right) R_2$,

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & 0 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

Applying $R_1 \to R_1 - 2R_2$,

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

$\therefore X = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

5. L.H.S.

$= ac\cos B - bc\cos A$

$= ac\left(\frac{c^2+a^2-b^2}{2ca}\right) - bc\left(\frac{b^2+c^2-a^2}{2bc}\right)$ ...[By cosine rule]

$= \left(\frac{c^2+a^2-b^2}{2}\right) - \left(\frac{b^2+c^2-a^2}{2}\right)$

$= \frac{c^2+a^2-b^2-b^2-c^2+a^2}{2}$

$= \frac{2a^2-2b^2}{2}$

$= \frac{2\left(a^2-b^2\right)}{2}$

$= a^2 - b^2 =$ R.H.S.

6. Let $\bar{a} = -3\hat{i} + 4\hat{j} - 2\hat{k},\ \bar{b} = \hat{i} + 2\hat{k},\ \bar{c} = \hat{i} - p\hat{j}$.

Since $\bar{a}, \bar{b}, \bar{c}$ are coplanar,

$\left[\bar{a} \quad \bar{b} \quad \bar{c}\right] = 0$

$\therefore \begin{vmatrix} -3 & 4 & -2 \\ 1 & 0 & 2 \\ 1 & -p & 0 \end{vmatrix} = 0$

$\therefore -3(0+2p) - 4(0-2) - 2(-p-0) = 0$

$\therefore -6p + 8 + 2p = 0$

$\therefore -4p = -8$

$\therefore p = \frac{-8}{-4} = 2$

7. $5\bar{a} + 3\bar{b} - 8\bar{c} = \bar{0}$

$\therefore 8\bar{c} = 5\bar{a} + 3\bar{b}$

$\therefore \bar{c} = \frac{5\bar{a}+3\bar{b}}{8}$

$\therefore \bar{c} = \frac{3\bar{b}+5\bar{a}}{3+5}$

$\therefore$ By the section formula,

point C divides the line segment AB internally in the ratio $3 : 5$.

8. $\sim(\sim p \wedge \sim q) \vee q$

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$\sim(\sim p \wedge \sim q)$	$\sim(\sim p \wedge \sim q) \vee q$
T	T	F	F	F	T	T
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	T	F	F

Truth values in the last column are not identical.

$\therefore \ \sim(\sim p \wedge \sim q) \vee q$ is a contingency.

9. Let $f(x)=\sqrt{x}$

$\therefore f'(x)=\frac{1}{2\sqrt{x}}$

$x=8.95=9-0.05=a+h$

Here, $a=9$ and $h=-0.05$

$f(a)=f(9)\doteq\sqrt{9}=3$ and

$f'(a)=f'(9)=\frac{1}{2\sqrt{9}}=\frac{1}{6}$

$f(a+h)\approx f(a)+hf'(a)$

$\therefore\sqrt{8.95}\approx 3+(-0.05)\left(\frac{1}{6}\right)$

$\approx 3-0.0083$

$\therefore\sqrt{8.95}\approx 2.9917$

10. Let $I=\int_{0}^{\frac{\pi}{2}}\sin^2 x\,dx=\int_{0}^{\frac{\pi}{2}}\frac{1-\cos 2x}{2}\,dx$

$=\frac{1}{2}\int_{0}^{\frac{\pi}{2}}(1-\cos 2x)dx$

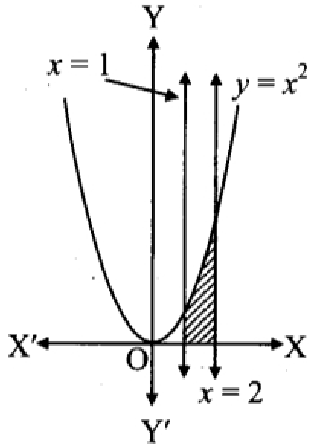
$=\frac{1}{2}\left[x-\frac{\sin 2x}{2}\right]_{0}^{\frac{\pi}{2}}$

$=\frac{1}{2}\left[\left(\frac{\pi}{2}-0\right)-\frac{1}{2}(\sin\pi-\sin 0)\right]$

$=\frac{1}{2}\left[\frac{\pi}{2}-\frac{1}{2}(0-0)\right]$

$\therefore I=\frac{\pi}{4}$

11. Required area



$=\int_{1}^{2}y\,dx$

$=\int_{1}^{2}x^2\,dx$

$=\frac{1}{3}\left[x^3\right]_{1}^{2}$

$=\frac{1}{3}\left(2^3-1^3\right)=\frac{1}{3}(8-1)$

$=\frac{7}{3}$ sq. units

12. $\frac{dy}{dx}+y=e^{-x}$

The given equation is of the form $\frac{dy}{dx}+Py=Q$,

where $P=1$ and $Q=e^{-x}$

$\therefore$ I.F. $=e^{\int P dx}=e^{\int 1.dx}=e^{x}$

$\therefore$ Solution of the given equation is

y (I.F.) $=\int Q(\text{ I.F. })dx+c$

$\therefore ye^{x}=\int e^{-x}e^{x}\,dx+c$

$\therefore ye^{x}=\int 1.\,dx+c$

$\therefore ye^{x}=x+c$

13. i. Since $f(x)$ is the probability density function of X,

$\int_0^2 f(x)dx=1$

$\therefore\int_0^2 kx\,dx=1$

$\therefore k\int_0^2 x dx=1$
$\therefore k\left[\frac{x^2}{2}\right]_0^2=1$
$\therefore k\left(\frac{2^2}{2}-0\right)=1$
$\therefore 2k=1$
$\therefore k=\frac{1}{2}$

ii. $P(1<X<2)=\int_1^2 f(x)dx$
$=\int_1^2 \frac{x}{2}\ dx$
$=\frac{1}{2}\int_1^2 x\ dx$
$=\frac{1}{2}\left[\frac{x^2}{2}\right]_1^2$
$=\frac{1}{4}(4-1)$
$\therefore P(1<X<2)=\frac{3}{4}$

14. Let $I=\int_0^{\frac{\pi}{2}}\sqrt{1-\cos 4x}dx$
$=\int_0^{\frac{\pi}{2}}\sqrt{2\sin^2 2x}\ dx$ ...$\left[\because 1-\cos A=2\sin^2\frac{A}{2}\right]$
$=\sqrt{2}\int_0^{\frac{\pi}{2}}\sin 2x dx$
$=\sqrt{2}\left[\frac{-\cos 2x}{2}\right]_0^{\frac{\pi}{2}}$
$=-\frac{\sqrt{2}}{2}(\cos\pi-\cos 0)$
$=-\frac{\sqrt{2}}{2}(-1-1)$
$\therefore I=\sqrt{2}$

Section C

15. $\sin x=\tan x$
$\therefore \sin x=\frac{\sin x}{\cos x}$
$\therefore \sin x\cos x-\sin x=0$
$\therefore \sin x(\cos x-1)=0$
$\therefore \sin x=0$ or $\cos x=1$
$\therefore \sin x=\sin 0$ or $\cos x=\cos 0$
Since, $\sin\theta=0$ implies $\theta=n\pi$ and $\cos\theta=\cos\alpha$
implies $\theta=2n\pi\pm\alpha, n\in Z$.
$\therefore x=n\pi$ or $x=2\ m\pi\pm 0$
$\therefore$ The required general solution is $x=n\pi$ or $x=2\ m\pi$, where $n, m\in Z$.

16. $a^2,\ b^2, c^2$ are in A.P. ...[given]
$\therefore a^2+c^2=2\ b^2$
$\therefore a^2+c^2-b^2=b^2$
$\therefore \frac{a^2+c^2-b^2}{2ac}=\frac{b^2}{2ac}$
$\therefore \cos\ B=\frac{k^2\sin^2\ B}{2siven]}$
... [By cosine and sine rule]
$\therefore \cos B=\frac{k^2\sin^2 B}{2k^2\sin A\sin C}$
$\therefore 2\cdot\frac{\cos B}{\sin B}=\frac{\sin B}{\sin A\sin C}$
$\therefore 2\cot B=\frac{\sin[\pi-(A+C)]}{\sin A\sin C}$ ...$[\because A+B+C=\pi]$
$=\frac{\sin(A+C)}{\sin A\sin C}$
$=\frac{\sin A\cos C+\cos A\sin C}{\sin A\sin C}$
$\therefore 2\cot B=\frac{\sin A\cos C}{\sin A\sin C}+\frac{\cos A\sin C}{\sin A\sin C}$
$=\cot C+\cot A$
Hence, $\cot A, \cot B, \cot C$ are in A.P.

17. Given equation is $kxy+10x+6y+4=0$
Comparing with
$ax^2+2hxy+by^2+2gx+2fy+c=0$, we get

$a=0,\ h=\frac{k}{2},\ b=0,\ g=5, f=3, c=4$

Since the given equation represents a pair of lines.

$abc+2fgh-af^2-bg^2-ch^2=0$

$(0)(0)(4)+2(3)(5)\left(\frac{k}{2}\right)-(0)(3)^2\ -(0)(5)^2-(4)\left(\frac{k}{2}\right)^2=0$

$\therefore 15k-k^2=0$

$\therefore -k(k-15)=0$

$\therefore k=0$ or $k=15$

If $k=0$, then the equation becomes $10x+6y+4=0$ which does not represent a pair of lines.

$\therefore k\neq 0, k=15$.

18. The acute angle θ between the planes $\bar{r}\cdot\bar{n}_1=p_1$ and $\bar{r}\cdot\bar{n}_2=p_2$ is given by

$\cos\theta=\left|\frac{\overline{n_1}\cdot\overline{n_2}}{|\overline{n_1}|\cdot|\cdot|\bar{n}_2}\right|$

Here, $\bar{n}_1=2\hat{i}+\hat{j}-\hat{k}$ and $\bar{n}_2=\hat{i}+2\hat{j}+\hat{k}$

$\therefore \bar{n}_1\cdot\bar{n}_2=(2\hat{\hat{i}}+\hat{j}-\hat{k})\cdot(\hat{i}+2\hat{j}+\hat{k})$

$=2(1)+1(2)+(-1)(1)$

$=2+2-1=3$

$|\bar{n}_1|=\sqrt{(2)^2+(1)^2+(-1)^2}=\sqrt{4+1+1}=\sqrt{6}$

$|\bar{n}_2|=\sqrt{(1)^2+(2)^2+(1)^2}=\sqrt{1+4+1}=\sqrt{6}$

$\therefore \cos\theta=\left|\frac{3}{\sqrt{6}\sqrt{6}}\right|=\frac{1}{2}$

$\therefore \cos\theta=\cos\left(\frac{\pi}{3}\right)$

$\therefore \theta=\frac{\pi}{3}$

Hence, the acute angle between the planes is $\frac{\pi}{3}$ i.e., 60°.

19. Let α,β,γ be the angles made by the line with positive directions of X, Y, Z axes respectively.

$\therefore \alpha=90^\circ, \beta=135^\circ, \gamma=45^\circ$

Let the direction cosines of the line be $l,\ m, n$

$\therefore l=\cos 90^\circ, m=\cos 135^\circ, n=\cos 45^\circ$

Now, $m=\cos 135^\circ=\cos(180^\circ-45^\circ)$

$=-\cos 45^\circ$

$=-\frac{1}{\sqrt{2}}$

$\therefore l=0,\ m=-\frac{1}{\sqrt{2}}, n=\frac{1}{\sqrt{2}}$

$\therefore$ Direction cosines of the line are $0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}$

20. Cartesian equation of the given line is

$3x-1=6y+2=1-z$

Dividing throughout by 6, we get

$\frac{3\left(x-\frac{1}{3}\right)}{6}=\frac{6\left(y+\frac{1}{3}\right)}{6}=\frac{-(z-1)}{6}$

$\therefore \frac{x-\frac{1}{3}}{2}=\frac{y+\frac{1}{3}}{1}=\frac{z-1}{-6}$

Let $\bar{a}$ be the position vector of the line passing through the point $\left(\frac{1}{3},\frac{-1}{3},1\right)$

$\therefore \bar{a}=\frac{1}{3}\hat{i}-\frac{1}{3}\hat{j}+\hat{k}$

direction ratios of the line are $2,1,-6$

Let $\bar{b}$ be the vector parallel to the line.

$\therefore \bar{b}=2\hat{i}+\hat{j}-6\hat{k}$

The vector equation of a line passing through a point with position vector $\bar{a}$ and parallel to $\bar{b}$ is $\bar{r}=\bar{a}+\lambda\bar{b}$

$\therefore$ Vector equation of the line is

$\bar{r}=\left(\frac{1}{3}\hat{i}-\frac{1}{3}\hat{j}+\hat{k}\right)+\lambda(2\hat{i}+\hat{j}-6\hat{k})$

21. Let $y=\cos^{-1}\left(\frac{3\cos x-2\sin x}{\sqrt{13}}\right)$

$\therefore y=\cos^{-1}\left(\frac{3}{\sqrt{13}}\cos x-\frac{2}{\sqrt{13}}\sin x\right)$

Put $\frac{3}{\sqrt{13}}=\cos A$ and $\frac{2}{\sqrt{13}}=\sin A$

Also, $\sin^2 A+\cos^2 A=\frac{4}{13}+\frac{9}{13}=1$

and $\tan A=\frac{2}{3}$

$\therefore A=\tan^{-1}\left(\frac{2}{3}\right)$

$\therefore y=\cos^{-1}(\cos x\cdot\cos\ A-\sin x\cdot\sin\ A)$

$=\cos^{-1}(\cos(x+A))$

$\therefore y=x+A=x+\tan^{-1}\left(\frac{2}{3}\right)$

Differentiating w.r.t. x, we get

$\frac{dy}{dx}=\frac{d}{dx}\left[x+\tan^{-1}\left(\frac{2}{3}\right)\right]=1+0$

$\therefore\frac{dy}{dx}=1$

22. Let $I=\int\frac{\sqrt{\tan x}}{\sin x\cdot\cos x}\ dx$

$=\int\frac{\frac{\sqrt{\tan x}}{\cos x}}{\frac{\sin x\cdot\cos x}{\cos^2 x}}\ dx$

$=\int\frac{\sec^2 x\sqrt{\tan x}}{\tan x}\ dx$

$=\int\frac{\sec^2 x}{\sqrt{\tan x}}\ dx$

Put $\tan x=t$

Differentiating w.r.t. x, we get

$\sec^2 x\ dx=dt$

$\therefore I=\int\frac{dt}{\sqrt{t}}=\int t^{\frac{-1}{2}}dt=\frac{t^{\frac{1}{2}}}{\frac{1}{2}}+c$

$\therefore I=2\sqrt{t}+c=2\sqrt{\tan x}+c$

23. $y=A\cos(\log x)+B\sin(\log x)$ ...(i)

Differentiating w.r.t. x, we get

$\frac{dy}{dx}=\frac{-A\sin(\log x)}{x}+\frac{B\cos(\log x)}{x}$

$\therefore x\frac{dy}{dx}=-A\sin(\log x)+B\cos(\log x)$

Again, differentiating w.r.t. x, we get

$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=\frac{-A\cos(\log x)}{x}-\frac{B\sin(\log x)}{x}$

$\therefore x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}\ =-[A\cos(\log x)+B\sin(\log x)]$

$\therefore x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=-y$ ...[From (i)]

$\therefore x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$

24. Let $f(x)=\sin x$

$\therefore f'(x)=\cos x$

$x=30^\circ30'=30^\circ+\left(\frac{1}{2}\right)^\circ=a+h$

Here, $a=30^\circ$

and $h=\left(\frac{1}{2}\right)^\circ=\frac{0.0175}{2}=0.00875$

$f(a)=f\left(30^\circ\right)=\sin 30^\circ=\frac{1}{2}=0.5$

$f'(a)=f'\left(30^\circ\right)=\cos 30^\circ=0.866$

$f(a+h)\approx f(a)+hf'(a)$

$\therefore\sin(30^\circ30')\approx 0.5+(0.00875)(0.866)$

$\approx 0.5+0.0075775$

$\therefore\sin(30^\circ30')\approx 0.5075775$

25. Let X be the number of bombs hitting the target.

P(bomb hits the target)$=p=0.8, q=1-0.8$

$=0.2$

Given, $n=10$

$\therefore X\sim B(10,0.8)$

The p.m.f. of X is given by

$P(X=x)=p(x)={ }^{10} C_x(0.8)^x(0.2)^{10-x}$,
$x=0,1,2, \ldots, 10$
$\therefore P$ (exactly 4 bombs will hit the target)
$=P(X=4)$
$={ }^{10} C_4(0.8)^4(0.2)^6$
$=\frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} \times 0.4096 \times 6.4 \times 10^{-5}$
$=210 \times 0.4096 \times 6.4 \times 10^{-5}$
$=550.5024 \times 10^{-5}$
$\therefore$ Required probability is 550.5024×10^{-5}.

26. X denote the number of sixes.
$\therefore$ Possible values of X are $0,1,2$.
Let $P($ getting six when a die is thrown $)=p=\frac{1}{6}$
$\therefore q=1-p=1-\frac{1}{6}=\frac{5}{6}$
$\therefore P(X=0)=P($ no six $)=qq=q^2=\left(\frac{5}{6}\right)^2=\frac{25}{36}$
$P(X=1)=P($ one six $)=pq+qp=2pq$
$=2\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)$
$=\frac{10}{36}$
$P(X=2)=P($ two sixes $)=pp=p^2=\left(\frac{1}{6}\right)^2=\frac{1}{36}$
Expectation of $X=E(X)$
$=\sum_{i=1}^3 x_i \cdot P\left(x_i\right)$
$=0 \times \frac{25}{36}+1 \times \frac{10}{36}+2 \times \frac{1}{36}$
$=\frac{1}{36}(0+10+2)=\frac{1}{3}$

Section D

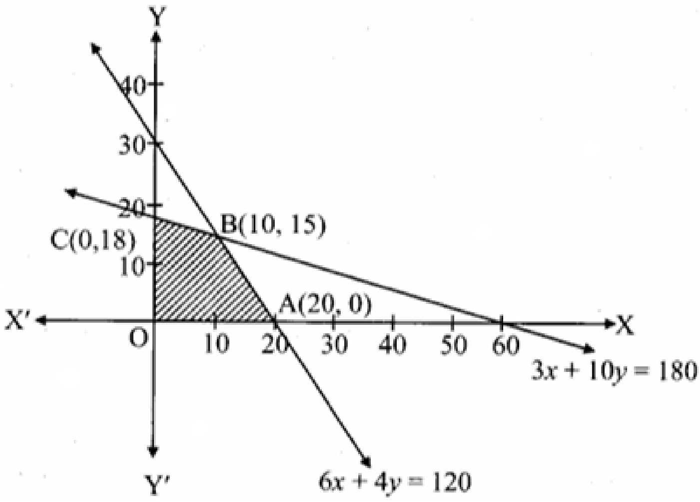
27. Let x number of bicycles and y number of tricycles be manufactured by the company.
$\therefore$ Total profit $Z=180x+220y$
This is the objective function to be maximized.
The given information can be tabulated as shown below:

	Bicycles (x)	Tricycles (y)	Maximum availability of time (hrs)
Machine A	6	4	120
Machine B	3	10	180

$\therefore$ The constraints are $6x+4y \leq 120, 3x+10y \leq 180, x \geq 0, y \geq 0$
$\therefore$ Given problem can be formulated as
Maximize: $Z=180x+220y$
Subject to: $6x+4y \leq 120, 3x+10y \leq 180, x \geq 0, y \geq 0$.
To draw the feasible region, construct the table as follows:

Inequality	$6x+4y \leq 120$	$3x+10y \leq 180$
Corresponding equation (of line)	$6x+4y=120$	$3x+10y=180$
Intersection of line with X -axis	$(20,0)$	$(60,0)$
Intersection of line with Y -axis	$(0,30)$	$(0,18)$
Region	Origin side	Origin side

Shaded portion OABC is the feasible region, whose vertices are $O \equiv(0,0), A \equiv(20,0), B$ and $C \equiv(0,18)$



B is the point of intersection of the lines $3x+10y=180$ and $6x+4y=120$.

Solving the above equations, we get

B $\equiv(10,15)$

Here the objective function is,

$Z=180x+220y$

$\therefore Z$ at $O(0,0)=180(0)+220(0)=0$

Z at $A(20,0)=180(20)+220(0)=3600$

Z at $B(10,15)=180(10)+220(15)=5100$

Z at $C(0,18)=180(0)+220(18)=3960$

$\therefore Z$ has maximum value 5100 at $B(10,15)$

$\therefore Z$ is maximum when $x=10, y=15$

Thus, the company should manufacture 10 bicycles and 15 tricycles to gain maximum profit of ₹ 5100.

28. Matrix form of the given system of equations is

$$\begin{bmatrix}1 & 1 & 1\\ 3 & -1 & 3\\ 5 & 5 & -4\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}6\\ 6\\ 3\end{bmatrix}$$

This is of the form $AX=B$,

where, $A=\begin{bmatrix}1 & 1 & 1\\ 3 & -1 & 3\\ 5 & 5 & -4\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}, B=\begin{bmatrix}6\\ 6\\ 3\end{bmatrix}$

Applying $R_2\rightarrow R_2-3R_1$ and $R_3\rightarrow R_3-5R_1$, we get

$$\begin{bmatrix}1 & 1 & 1\\ 0 & -4 & 0\\ 0 & 0 & -9\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}6\\ -12\\ -27\end{bmatrix}$$

Hence, the original matrix has reduced to upper triangular matrix.

Now by equality of matrices, we have

$x+y+z=6$ ...(i)

$-4y=-12$

i.e., $y=3$ ...(ii)

$-9z=-27$

i.e., $z=3$ ...(iii)

Now substituting $y=3, z=3$ in equation (i),

we get

$x+3+3=6$

$\therefore x+6=6$

$\therefore x=6-6$

$\therefore x=0$

Hence, the solution of given equations are $x=0, y=3, z=3$

29. Let $\bar{a}, \bar{b}, \bar{c}, \vec{d}$ be the position vectors of points A, B, C, D respectively.

$\therefore \bar{a}=2\hat{i}+\hat{j}-\hat{k}, \bar{b}=-\hat{j},\ \ \bar{c}=4\hat{i}+4\hat{k}, \bar{d}=2\hat{i}+\hat{k}$

$\therefore \overline{AB}=\bar{b}-\bar{a}=(-\hat{j})-(2\hat{i}+\hat{j}-\hat{k})$

$= -2\hat{i} - 2\hat{j} + \hat{k}$
$\overline{AC} = \bar{c} - \bar{a} = (4\hat{i} + 4\hat{k}) - (2\hat{i} + \hat{j} - \hat{k})$
$= 2\hat{i} - \hat{j} + 5\hat{k}$
$\overline{AD} = \bar{d} - \bar{a} = (2\hat{i} + \hat{k}) - (2\hat{i} + \hat{j} - \hat{k})$
$= -\hat{j} + 2\hat{k}$
Points A, B, C, D are coplanar if $\overline{AB}, \overline{AC}$ and $\overline{AD}$ are coplanar.
Let $\overline{AB}, \overline{AC}$ and $\overline{AD}$ be coplanar.
$\therefore$ There exist unique scalars t_1 and t_2 such that
$\overline{AB} = t_1\overline{AC} + t_2\overline{AD}$
$\therefore -2\hat{i} - 2\hat{j} + \hat{k} = t_1(2\hat{i} - \hat{j} + 5\hat{k}) + t_2(-\hat{j} + 2\hat{k})$
$\therefore -2\hat{i} - 2\hat{j} + \hat{k} = (2t_1)\,\hat{i} + (-t_1 - t_2)\,\hat{j} + (5t_1 + 2t_2)\,\hat{k}$
$\therefore$ By equality of vectors, we get
$2t_1 = -2$, i.e., $t_1 = -1$
$-t_1 - t_2 = -2 \Rightarrow 1 - t_2 = -2 \Rightarrow t_2 = 3$
$5t_1 + 2t_2 = 1$
$\Rightarrow$ L.H.S. $= 5(-1) + 2(3) = 1 =$ R.H.S.
$\therefore t_1$ and t_2 are unique.
$\therefore \overline{AB}, \overline{AC}, \overline{AD}$ are coplanar.
$\therefore$ Point A, B, C, D are also coplanar.

30. Let $\bar{a}, \bar{b}, \bar{c}$ be the position vectors of the points A, B and C respectively.
$\therefore \bar{a} = \hat{i} + \hat{k}, \bar{b} = \hat{i} - \hat{j} + \hat{k},\ \ \bar{c} = 4\hat{i} - 3\hat{j} + 2\hat{k}$
The vector equation of the plane passing through the points $A(\bar{a}), B(\bar{b})$ and $C(\bar{c})$ is
$\bar{r} \cdot (\overline{AB} \times \overline{AC}) = \bar{a} \cdot (\overline{AB} \times \overline{AC})$
Now, $\overline{AB} = \bar{b} - \bar{a}\ = (\hat{i} - \hat{j} + \hat{k}) - (\hat{i} + \hat{k}) = -\hat{j}$
and $\overline{AC} = \bar{c} - \bar{a}\ = (4\hat{i} - 3\hat{j} + 2\hat{k}) - (\hat{i} + \hat{k})$
$= 3\hat{i} - 3\hat{j} + \hat{k}$
$$\therefore \overline{AB} \times \overline{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -1 & 0 \\ 3 & -3 & 1 \end{vmatrix}$$
$= \hat{i}(-1 - 0) - \hat{j}(0) + \hat{k}(0 + 3)$
$= -\hat{i} + 3\hat{k}$
$\therefore$ Vector equation of the required plane is
$\bar{r} \cdot (-\hat{i} + 3\hat{k}) = (\hat{i} + \hat{k}) \cdot (-\hat{i} + 3\hat{k})$
$\therefore \bar{r}(-\hat{i} + 3\hat{k}) = 1(-1) + 1(3)$
$\therefore \bar{r}(-\hat{i} + 3\hat{k}) = -1 + 3$
$\therefore \bar{r}(-\hat{i} + 3\hat{k}) = 2$

31. Proof:
x and y are differentiable functions of t.
Let there be a small increment δt in the value of t.
Correspondingly, there should be a small increments $\delta x, \delta y$ in the values of x and y respectively.
As $\delta t \to 0, \delta x \to 0, \delta y \to 0$
Consider, $\frac{\delta y}{\delta x} = \frac{\frac{\delta y}{\delta t}}{\frac{\delta x}{\delta t}}$
Taking $\lim\limits_{\delta \to 0}$ on both sides, we get
$$\lim_{\delta \to 0} \frac{\delta y}{\delta x} = \frac{\lim\limits_{\delta t \to 0} \frac{\delta y}{\delta t}}{\lim\limits_{\delta t \to 0} \frac{\delta x}{\delta t}}$$
Since x and y are differentiable functions of t, $\lim\limits_{\delta \to 0} \frac{\delta y}{\delta t} = \frac{dy}{dt}$ exists and is finite.
$\lim\limits_{\delta \to 0} \frac{\delta x}{\delta t} = \frac{dx}{dt}$ exists and is finite.
$\therefore \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)$ (as $\delta t \to 0, \delta x \to 0$)

$\therefore$ Limits on right hand side exists and are finite.

$\therefore$ Limits on the left hand side should also exists and be finite.

$\therefore \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$ exists and is finite.

$\therefore \frac{dy}{dx} = \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right), \frac{dx}{dt} \neq 0$

$x = a\cos t, y = a\sin t$

$\therefore \frac{dx}{dt} = -a\sin t$ and $\frac{dy}{dt} = a\cos t$

$\therefore \frac{dy}{dx} = \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = -\frac{a\cos t}{a\sin t} = -\cot t$

$x = a\cos^2 t$ and $y = a\sin^2 t$

$\therefore \frac{dx}{dt} = \frac{d}{dt}\left(a\cos^2 t\right) = a[2\cos t(-\sin t)]$

$\therefore \frac{dx}{dt} = -2a\sin t\cos t$

and $\frac{dy}{dt} = \frac{d}{dt}\left(a\sin^2 t\right)$

$\therefore \frac{dy}{dt} = a(2\sin t\cos t) = 2a\sin t\cos t$

$\therefore \frac{dy}{dx} = \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \frac{2a\sin t\cos t}{-2a\sin cost} = -1$

$x = at^2$

$\therefore \frac{dx}{dt} = \frac{d}{dt}\left(at^2\right) = a\frac{d}{dt}\left(t^2\right) = 2at$

$y = 2at$

$\therefore \frac{dy}{dt} = \frac{d}{dt}(2at) = 2a\frac{d}{dt}(t) = 2a$

$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{2a}{2at}$

$\therefore \frac{dy}{dx} = \frac{1}{t}$

32. $f(x) = x^2 - 5x + 9, x \in [1, 4]$

As $f(x)$ is a polynomial function, it is continuous and differentiable everywhere on its domain. Thus,

1. $f(x)$ is continuous on $[1, 4]$
2. $f(x)$ is differentiable on $(1, 4)$

$f(1) = (1)^2 - 5(1) + 9 = 1 - 5 + 9 = 5$

$f(4) = (4)^2 - 5(4) + 9 = 16 - 20 + 9 = 5$

$\therefore f(1) = f(4)$

Thus, all the conditions of Rolle's theorem are satisfied.

The derivative of $f(x)$ should vanish for at least one point $c \in [1, 4]$

To obtain the value of c,

$f(x) = x^2 - 5x + 9$

$\therefore f'(x) = 2x - 5$

$f'(c) = 0$

$\therefore 2c - 5 = 0$

$\therefore c = \frac{5}{2}$

$c = \frac{5}{2}$ lies between 1 and 4.

Thus, Rolle's theorem is verified.

33. Proof:

Let $I = \int \sqrt{a^2 - x^2} \cdot 1\, dx$

$= \sqrt{a^2 - x^2} \int 1\, dx - \int \left[\frac{d}{dx}\left(\sqrt{a^2 - x^2}\right) \cdot \int 1\, dx\right] dx$

$= \sqrt{a^2 - x^2} \cdot x - \int \frac{-2x}{2\sqrt{a^2 - x^2}} \cdot x\, dx$

$= x \cdot \sqrt{a^2 - x^2} - \int \frac{\left(a^2 - x^2\right) - a^2}{\sqrt{a^2 - x^2}}\, dx$

$= x \cdot \sqrt{a^2 - x^2} - \int \left(\frac{a^2 - x^2}{\sqrt{a^2 - x^2}} - \frac{a^2}{\sqrt{a^2 - x^2}}\right) dx$

$= x \cdot \sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}\, dx + a^2 \int \frac{1}{\sqrt{a^2 - x^2}}\, dx$

$\therefore I = x \cdot \sqrt{a^2 - x^2} - I + a^2 \sin^{-1}\left(\frac{x}{a}\right) + c_1$

$\therefore 2I = x \cdot \sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right) + c_1$

$\therefore I = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{c_1}{2}$

$\therefore \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$,

where $c = \frac{c_1}{2}$

34. Let $I = \int_0^{\pi} \frac{x}{a^2 \cos^2 x + b^2 \sin^2 x}\, dx$ ...(i)

$\therefore I = \int_0^{\pi} \frac{\pi - x}{a^2 \cos^2(\pi - x) + b^2 \sin^2(\pi - x)} dx$

...$\left[\because \int_0^2 \mathrm{f}(x)\mathrm{d}x = \int_0^1 \mathrm{f}(\mathrm{a} - x)\mathrm{d}x\right]$

$\therefore I = \int_0^{\pi} \frac{\pi - x}{a^2 \cos^2 x + b^2 \sin^2 x}\, dx$ ...(ii)

Adding (i) and (ii), we get

$2I = \int_0^{\pi} \frac{x + \pi - x}{a^2 \cos^2 x + b^2 \sin^2 x}\, dx$

$= \pi \int_0^{\pi} \frac{1}{a^2 \cos^2 x + b^2 \sin^2 x}\, dx$

Dividing Nr and Dr by $\cos^2 x$, we get

$2I = \pi \int_0^{\pi} \frac{\sec^2 x}{a^2 + b^2 \tan^2 x}\, dx$

$= \pi \int_0^{\frac{\pi}{2}} \left[\frac{\sec^2 x}{a^2 + b^2 \tan^2 x} + \frac{\sec^2(\pi - x)}{a^2 + b^2 \tan^2(\pi - x)}\right] dx$

...$\left[\because \int_0^{24} \mathrm{f}(x)\mathrm{d}x = \int_0^2 [\mathrm{f}(x) + \mathrm{f}(2\mathrm{a} - x)]\mathrm{d}x\right]$

$= \pi \int_0^{\frac{\pi}{2}} \left[\frac{\sec^2 x}{a^2 + b^2 \tan^2 x} + \frac{\sec^2 x}{a^2 + b^2 \tan^2 x}\right] dx$

$= 2\pi \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{a^2 + b^2 \tan^2 x}\, dx$

$\therefore I = \pi \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{a^2 + b^2 \tan^2 x}\, dx$

Put $\tan x = t$

$\therefore \sec^2 x\, dx = dt$

When $x = 0, t = 0$ and when $x = \frac{\pi}{2}, t = \infty$

$\therefore I = \pi \int_0^{\infty} \frac{dt}{a^2 + b^2 t^2}$

$= \frac{\pi}{b^2} \int_0^{\infty} \frac{dt}{\left(\frac{a}{b}\right)^2 + t^2}$

$= \frac{\pi}{b^2} \times \frac{1}{\left(\frac{a}{b}\right)} \left[\tan^{-1}\left(\frac{t}{\frac{a}{b}}\right)\right]_0^{\infty}$

$= \frac{\pi}{ab}\left[\tan^{-1}\left(\frac{bt}{a}\right)\right]_0^{\infty}$

$= \frac{\pi}{ab}\left(\tan^{-1}\infty - \tan^{-1} 0\right)$

$= \frac{\pi}{ab}\left(\frac{\pi}{2} - 0\right)$

$\therefore I = \frac{\pi^2}{2ab}$