Solution

MATHEMATICS

Class 12 - Maths & Stats (Gen)

Section A

1. Select and write the correct answer for the following multiple-choice type of questions :

(i) **(c)** $p \rightarrow (q \wedge \sim r)$

Explanation: {

$\sim [p \wedge (q \rightarrow r)]$

$\equiv \sim p \vee \sim (q \rightarrow r)$

$\equiv p \rightarrow [\sim (q \rightarrow r)] \ldots [\because p \rightarrow q \equiv \sim p \vee q]$

$\equiv p \rightarrow [\sim (\sim q \vee r)]$

$\equiv p \rightarrow (q \wedge \sim r)$

(ii) **(a)** $\frac{1}{\sqrt{10}}$

Explanation: {

Here, we have

$a = 18,\ b = 24, c = 30$

$\therefore\ s = \frac{a+b+c}{2}$

$s = \frac{18+24+30}{2} = \frac{72}{2} = 36$

$\therefore \sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$

$= \sqrt{\frac{(36-24)(36-30)}{24 \times 30}} = \sqrt{\frac{12 \times 6}{24 \times 30}} = \sqrt{\frac{1}{10}}$

$\therefore \sin \frac{A}{2} = \frac{1}{\sqrt{10}}$

(iii) **(d)** ± 1

Explanation: {

Here we have the combined equation as $12x^2 + bxy - y^2 = 0$

Comparing this equation with standard equation $ax^2 + 2hxy + by^2 = 0$, we get

$a = 12, 2h = b, b = -1$

$\therefore$ Let m_1 and m_2 be the slopes of the lines represented by given combined equation.

$\therefore m_1 + m_2 = \frac{-2h}{b} = b$ and $m_1 m_2 = \frac{a}{b} = -12$

$\therefore (m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1 m_2$

$(7)^2 = b^2 - 4(-12) \ldots [\because$ given $m_1 - m_2 = 7]$

$\therefore 49 = b^2 + 48$

$\therefore\ b^2 = 49 - 48 = 1$

$\therefore\ b = \pm 1$

(iv) **(a)** $(\sim p \wedge q) \vee \sim r$

Explanation: {

$\sim [(p \vee \sim q) \wedge r]$

$\equiv \sim (p \vee \sim q) \vee \sim r \ldots$ [De Morgan's law]

$\equiv (\sim p \wedge q) \vee \sim r \ldots$ [De Morgan's law]

(v) **(c)** 0

Explanation: {

$\sec\left(\frac{x+y}{x-y}\right) = a^2$

$\therefore \frac{x+y}{x-y} = \sec^{-1}\left(a^2\right) = b$ (say)

$\therefore x + y = bx - by$

$\therefore (1 + b)y = (b - 1)x$

$\therefore y = \left(\frac{b-1}{b+1}\right) x$

$\therefore y = cx$, where $c = \frac{b-1}{b+1} = \frac{y}{x}$ ...(i)

Differentiating w.r.t. x, we get

$\frac{dy}{dx}=c$

$\therefore \frac{dy}{dx}=\frac{y}{x}$ ...(ii)[From (I)]

$\frac{d^2y}{dx^2}=\frac{x\frac{dy}{dx}-y.1}{x^2}$

$=\frac{x\left(\frac{y}{x}\right)-y}{x^2}\ \ldots$[From (ii)]

$=\frac{y-y}{x^2}=0$

(vi) **(d)** $\tan\left(\frac{x}{2}\right)+c$

Explanation: {

$\int\frac{1}{1+\cos x}dx=\int\frac{1}{2\cos^2\frac{x}{2}}\ dx$

$=\frac{1}{2}\int\sec^2\frac{x}{2}\ dx$

$=\frac{1}{2}\cdot\frac{\tan\frac{x}{2}}{\frac{1}{2}}+c$

$=\tan\frac{x}{2}+c$

(vii) **(d)** $\sec x+\tan x$

Explanation: {

$\frac{dy}{dx}+y\sec x=\tan x$

This is of the form $\frac{dy}{dx}+Py=Q$, where $P=\sec x$

$\therefore$ I.F. $=e^{\int Pdx}=e^{\int\sec x dx}=e^{\log|\sec x+\tan x|}$

$\therefore$ I.F. $=\sec x+\tan x$

(viii) **(b)** 2.4

Explanation: {

$E(X)=np$

$\therefore 6=n(0.6)$

$\therefore n=10$

$q=1-p=1-0.6=0.4$

$\mathrm{Var}(X)=npq$

$=(10)(0.6)(0.4)$

$=2.4$

2. Answer the following questions :

(i) If $h^2=ab$, then the lines represented by the equation $ax^2+2\ hxy+by^2=0$ are coincident.

$\therefore$ The slopes of the lines will be equal.

$\therefore$ The ratio of their slopes $=1:1$

(ii) $5\bar{a}-3\bar{b}-2\bar{c}=\bar{0}$

$\therefore 2\bar{c}=5\bar{a}-3\bar{b}$

$\therefore \bar{c}=\frac{5\bar{a}-3\bar{b}}{2}$

$\therefore \bar{c}=\frac{5\bar{a}-3\bar{b}}{5-3}$

This shows that the point C divides BA externally in the ratio 5:3.

(iii) Let $I=\int\frac{dx}{x^2+4x+8}$

$\left(\frac{1}{2}\text{ coefficient of }x\right)^2=\left(\frac{1}{2}\times 4\right)^2=(2)^2=4$

$\therefore I=\int\frac{dx}{x^2+4x+4-4+8}=\int\frac{dx}{x^2+4x+4+4}$

$=\int\frac{1}{(x+2)^2+2^2}dx$

$\therefore I=\frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right)+c$

(iv) $(y''')^2+3\left(y''\right)+3xy'+5y=0$

Here, the highest order derivative is (y''') i.e. $\frac{d^3y}{dx^3}$ with power 2.

By definition of order and degree,

order : 3 and degree : 2

Section B

3. $(p \to q) \leftrightarrow (\sim p \vee q)$

p	q	$p \to q$	$\sim p$	$-p \vee q$	$(p \to q) \leftrightarrow (\sim p \vee q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

All the truth values in the last column are T. Hence, it is a tautology.

4. Let $A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore |A| = \cos\theta(0-0) - (-\sin\theta)(0-0) + 0(0-0)$

$= 0 + 0 + 0$

$\therefore |A| = 0$

$\therefore A$ is a singular matrix.

$\therefore$ Inverse of the given matrix does not exist.

5. Let $x = \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

$\therefore \sin x = -\frac{1}{\sqrt{2}}$

$\therefore \sin x = \sin\left(-\frac{\pi}{4}\right)$

$\therefore x = \frac{-\pi}{4}$

The principal value of $\sin^{-1}$ is $\left[-\frac{\pi}{2} \frac{\pi}{2}\right]$ and $-\frac{\pi}{2} \leq -\frac{\pi}{4} \leq \frac{\pi}{2}$.

$\therefore x = \frac{-\pi}{4}$

6. Let $\bar{a} = 2\hat{i} + 2\hat{j} - \hat{k}$

$\therefore |\bar{a}| = \sqrt{2^2 + 2^2 + (-1)^2}$

$= \sqrt{4+4+1} = \sqrt{9} = 3$

$\therefore \hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{2\hat{i}+2\hat{j}-\hat{k}}{3} = \frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}$

$\therefore$ The direction cosines of a are $\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}$.

7. $5\bar{a} + 3\bar{b} - 8\bar{c} = \bar{0}$

$\therefore 8\bar{c} = 5\bar{a} + 3\bar{b}$

$\therefore \bar{c} = \frac{5\bar{a}+3\bar{b}}{8}$

$\therefore \bar{c} = \frac{3\bar{b}+5\bar{a}}{3+5}$

$\therefore$ By the section formula,

point C divides the line segment AB internally in the ratio $3:5$.

8. L.H.S.

$= (p \wedge q) \vee (\sim p \wedge q) \vee (p \wedge \sim q)$

$\equiv (p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge q)$ ...[Commutative law]

$\equiv [p \wedge (q \vee \sim q)] \vee (\sim p \wedge q)$ ...[Distributive law]

$\equiv (p \wedge T) \vee (\sim p \wedge q)$ ...[Complement law]

$\equiv p \vee (\sim p \wedge q)$ ...[Identity law]

$\equiv (p \vee \sim p) \wedge (p \vee q)$ ...[Distributive law]

$\equiv T \wedge (p \vee q)$ ...[Complement law]

$\equiv p \vee q$ ...[Identity law]

$=$ R.H.S

9. Let $f(x) = \cos x$

$\therefore f'(x) = -\sin x$

$x = 60^\circ 30' = 60^\circ + \frac{1^\circ}{2} = a + h$

Here, $a = 60^\circ$

and $h = \frac{1}{2}^{\circ} = \frac{0.0175}{2} = 0.00875$

$f(a) = f(60^{\circ}) = \cos(60^{\circ}) = \frac{1}{2} = 0.5$

$f'(a) = f'(60^{\circ}) = -\sin(60^{\circ}) = -0.8660$

$f(a+h) \approx f(a) + hf'(a)$

$\therefore \cos(60^{\circ}30') \approx 0.5 + (0.00875)(-0.8660)$

$\approx 0.5 - 0.0075775$

$\therefore \cos(60^{\circ}30') \approx 0.4924$

10. Let $I = \int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x}\, dx$

$= \int_0^{\frac{\pi}{2}} \frac{1}{2\cos^2\left(\frac{x}{2}\right)} dx$

$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \sec^2\left(\frac{x}{2}\right) dx$

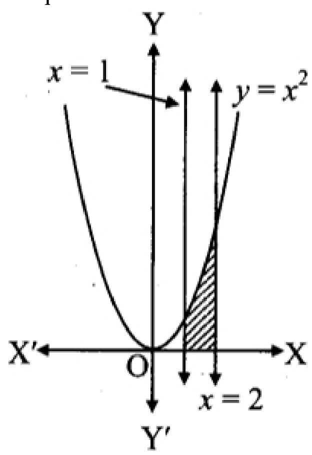
$= \frac{1}{2} \left[\frac{\tan\left(\frac{x}{2}\right)}{\frac{1}{2}}\right]_0^{\frac{\pi}{2}}$

$= \tan\frac{\pi}{4} - \tan 0$

$= 1 - 0$

$\therefore$ I = 1

11. Required area



$= \int_1^2 y dx$

$= \int_1^2 x^2\, dx$

$= \frac{1}{3}\left[x^3\right]_1^2$

$= \frac{1}{3}\left(2^3 - 1^3\right) = \frac{1}{3}(8-1)$

$= \frac{7}{3}$ sq. units

12. Let s be the displacement of the particle at time 't'.

Its velocity and acceleration are $\frac{ds}{dt}$ and $\frac{d^2 s}{dt^2}$ respectively.

According to the given condition,

$\frac{d^2 s}{dt^2} \propto \frac{ds}{dt}$

$\therefore \frac{d^2 s}{dt^2} = k\frac{ds}{dt}$, (where k is constant and $k \neq 0$)

13. $E(X) = \sum x_i \cdot P(x_i)$

$= 1\left(\frac{1}{5}\right) + 2\left(\frac{2}{5}\right) + 3\left(\frac{2}{5}\right)$

$= \frac{1+4+6}{5}$

$= \frac{11}{5} = 2.2$

$E\left(X^2\right) = \sum x_i^2 \cdot P(x_i)$

$= 1^2\left(\frac{1}{5}\right) + 2^2\left(\frac{2}{5}\right) + 3^2\left(\frac{2}{5}\right)$

$= \frac{1+8+18}{5}$
$= \frac{27}{5} = 5.4$
$\therefore \operatorname{Var}(X) = E\left(X^2\right) - [E(X)]^2$
$= 5.4 - (2.2)^2$
$= 5.4 - 4.84 = 0.56$
$\therefore \sigma = \sqrt{\operatorname{Var}(X)} = \sqrt{0.56} = 0.7483$

14. Let $I = \int_0^{\frac{\pi}{2}} \sqrt{1-\cos 4x}\,dx$
$= \int_0^{\frac{\pi}{2}} \sqrt{2\sin^2 2x}\ dx$ ...$\left[\because 1-\cos A = 2\sin^2 \frac{A}{2}\right]$
$= \sqrt{2}\int_0^{\frac{\pi}{2}} \sin 2x dx$
$= \sqrt{2}\left[\frac{-\cos 2x}{2}\right]_0^{\frac{\pi}{2}}$
$= -\frac{\sqrt{2}}{2}(\cos\pi - \cos 0)$
$= -\frac{\sqrt{2}}{2}(-1-1)$
$\therefore I = \sqrt{2}$

Section C

15. $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$
$\therefore \tan^{-1}\left(\frac{2x+3x}{1-(2x)(3x)}\right) = \frac{\pi}{4}$ ...$\left(\tan^{-1}\ A + \tan^{-1}\ B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)\right)$
$\therefore \frac{5x}{1-6x^2} = \tan\frac{\pi}{4} = 1$
$\therefore 5x = 1-6x^2$
$\therefore 6x^2 + 5x - 1 = 0$
$\therefore 6x^2 + 6x - x - 1 = 0$
$\therefore 6x(x+1) - 1(x+1) = 0$
$\therefore (x+1)(6x-1) = 0$
$\therefore x = -1$ or $x = \frac{1}{6}$
But $x = -1$ does not satisfy $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$
$\therefore x = \frac{1}{6}$

16. We know that, $1-\cos A = 2\sin^2\frac{A}{2}$
$\therefore 1-\left(\frac{b^2+c^2-a^2}{2bc}\right) = 2\sin^2\frac{A}{2} \ldots$ [By cosine rule]
$\therefore \frac{2bc-b^2-c^2+a^2}{2bc} = 2\sin^2\frac{A}{2}$
$\therefore \frac{a^2-\left(b^2-2bc+c^2\right)}{2bc} = 2\sin^2\frac{A}{2}$
$\therefore \frac{a^2-(b-c)^2}{2bc} = 2\sin^2\frac{A}{2}$
$\therefore \frac{(a+b-c)(a-b+c)}{2bc} = 2\sin^2\frac{A}{2}$
$\therefore \frac{(a+b+c-2c)(a+b+c-2b)}{2bc} = 2\sin^2\frac{A}{2}$
$\therefore \frac{(2s-2c)(2s-2b)}{2bc} = 2\sin^2\frac{A}{2}$
...$[\because a+b+c = 2s($ given $)]$
$\therefore \frac{(s-c)(s-b)}{bc} = \sin^2\frac{A}{2}$
$\therefore \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$
...[As $\angle\frac{A}{2}$ is an acute angle $\therefore \sin\frac{A}{2} > 0]$
Similarly, $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$
and $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$ can be proved.

17. Auxiliary equation of $ax^2 + 2hxy + by^2 = 0$ is $bm^2 + 2hm + a = 0$.
Since one of the lines bisects an angle between the co-ordinate axes, the line will make an angle of 45° or 135° with the positive direction of X-axis.
$\therefore$ Slope of the line $= \tan 45^\circ$ or $\tan 135^\circ$
$\therefore m = \tan 45^\circ$ or $\tan(180^\circ - 45^\circ) = -\tan 45^\circ$
$\therefore m = 1$ or -1

$\therefore m=\pm 1$ are the roots of the auxiliary equation $bm^2+2hm+a=0$.

$\therefore b(\pm 1)^2+2h(\pm 1)+a=0$

$\therefore b\pm 2h+a=0$

$\therefore a+b=\mp 2h$

$\therefore (a+b)^2=4h^2$

18. Here we have the points as $A(3,4,-7)$ and $B(6,-1,1)$

$\therefore$ The cartesian equation of the line passing through A and B is

$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$

$\therefore \frac{x-3}{6-3}=\frac{y-4}{-1-4}=\frac{z+7}{1-(-7)}$

$\therefore \frac{x-3}{3}=\frac{y-4}{-5}=\frac{z+7}{8}$

for vector form of the equation:

Let $\frac{x-3}{3}=\frac{y-4}{-5}=\frac{z+7}{8}=\lambda$

$\therefore x=3\lambda+3, y=-5\lambda+4, z=8\lambda-7$

Now the equation in vector form is

$\bar{r}=x\hat{i}+y\hat{j}+z\hat{k}$

$\bar{r}=(3\lambda+3)\hat{i}+(-5\lambda+4)\hat{j}+(8\lambda-7)\hat{k}$

$=(3\hat{i}+4\hat{j}-7\hat{k})+\lambda(3\hat{i}-5\hat{j}+8\hat{k})$

$\therefore \bar{r}=(3\hat{i}+4\hat{j}-7\hat{k})+\lambda(3\hat{i}-5\hat{j}+8\hat{k})$

19. Given that $2\bar{a}+3\bar{b}-5\bar{c}=0$

$\therefore 2\bar{a}+3\bar{b}=5\bar{c}$

$\therefore \bar{c}=\frac{2\bar{a}+3\bar{b}}{5}$

$\therefore \bar{c}=\frac{3\bar{a}+2\bar{b}}{3+2}$

Now if we assume that C divides line segment internally AB in the ratio $m:n$, then we have

$\bar{c}=\frac{m\bar{b}+n\bar{a}}{m+n}$

$\Rightarrow m=3$ and $n=2$

Hence, C divides line segment AB internally in the ratio $3:2$

20. The equation of the plane passing through the intersection of the planes $3x+2y-z+1=0$ and $x+y+z-2=0$ is

$(3x+2y-z+1)+\lambda\cdot(x+y+z-2)=0$ ...(i)

But the plane passes through $(2,2,1)$,

$\therefore (6+4-1+1)+\lambda\cdot(2+2+1-2)=0$

$\therefore 10+3\lambda=0$

$\therefore \lambda=\frac{-10}{3}$

Putting the value of λ in (i), we get

$(3x+2y-z+1)-\frac{10}{3}(x+y+z-2)=0$

$\therefore 9x+6y-3z+3-10x-10y-10z+20=0$

$\therefore -x-4y-13z+23=0$

$\therefore x+4y+13z-23=0$

which is the equation of the required plane.

21. $x^py^q=(x+y)^{p+q}$

Taking log on both sides, we get

$\log(x^py^q)=\log(x+y)^{p+q}$

$\therefore \log x^p+\log y^q=(p+q)\log(x+y)$

$\therefore p\log x+q\log y=(p+q)\log(x+y)$

Differentiating w.r.t. x, we get

p. $\frac{1}{x}+q\cdot\frac{1}{y}\cdot\frac{dy}{dx}=(p+q)\cdot\frac{1}{x+y}\cdot\frac{d}{dx}(x+y)$

$\therefore \frac{p}{x}+\frac{q}{y}\cdot\frac{dy}{dx}=\frac{p+q}{x+y}\left(1+\frac{dy}{dx}\right)$

$\therefore \frac{q}{y}\cdot\frac{dy}{dx}-\left(\frac{p+q}{x+y}\right)\frac{dy}{dx}=\frac{p+q}{x+y}-\frac{p}{x}$

$\therefore \frac{dy}{dx}\left(\frac{q}{y}-\frac{p+q}{x+y}\right)=\frac{(p+q)x-p(x+y)}{x(x+y)}$

$\therefore \frac{dy}{dx}\left(\frac{qx+qy-py-qy}{y(x+y)}\right)=\frac{px+qx-px-py}{x(x+y)}$

$\therefore \frac{dy}{dx}\left(\frac{qx-py}{y}\right)=\frac{qx-py}{x}$

$\therefore \frac{dy}{dx}\left(\frac{1}{y}\right)=\frac{1}{x}$

$\therefore \frac{dy}{dx}=\frac{y}{x}$

22. Let $I=\int \sec^3 2x\ dx$

$=\int \sec 2x \cdot \sec^2 2x\ dx$

$=\sec 2x \int \sec^2 2x\ dx\ -\int\left[\frac{d}{dx}(\sec 2x) \int \sec^2 2x\ dx\right] dx$

$=\sec 2x \cdot \frac{\tan 2x}{2}-\int(2 \sec 2x \tan 2x) \cdot \frac{\tan 2x}{2}\ dx$

$=\frac{1}{2} \sec 2x \tan 2x-\int \sec 2x \tan^2 2x\ dx$

$=\frac{1}{2} \sec 2x \tan 2x-\int \sec 2x\left(\sec^2 2x-1\right) dx$

$=\frac{1}{2} \sec 2x \tan 2x-\int \sec^3 2x dx+\int \sec 2x dx$

$\therefore I=\frac{1}{2} \sec 2x \tan 2x-I+\frac{1}{2} \log |\sec 2x+\tan 2x|+c_1$

$\therefore .2I=\frac{1}{2} \sec 2x \tan 2x+\frac{1}{2} \log |\sec 2x+\tan 2x|+c_1$

$\therefore I=\frac{1}{4} \sec 2x \tan 2x+\frac{1}{4} \log |\sec 2x+\tan 2x|+c$, where $c=\frac{c_1}{2}$

23. $\frac{dy}{dx}+y \sec x=\tan x$

The given equation is of the form

$\frac{dy}{dx}+Py=Q$,

where $P=\sec x$ and $Q=\tan x$

$\therefore$ I.F. $=e^{\int P dr}$

$=e^{\int \sec x dx}$

$=e^{\log |\sec x+\tan x|}$

$=\sec x+\tan x$

$\therefore$ Solution of the given equation is

$y(\text{ I.F. })=\int Q(\text{ I.F. }) dx+c$

$y(\sec x+\tan x)=\int \tan x(\sec x+\tan x) dx+c$

$=\int\left(\sec x \tan x+\tan^2 x\right) dx+c$

$=\int\left(\sec x \tan x+\sec^2 x-1\right) dx+c$

$\therefore y(\sec x+\tan x)=\sec x+\tan x-x+c$

24. Equation of the curve is $y=x^2+2e^x+2$

Differentiating w.r.t. x, we get

$\frac{dy}{dx}=2x+2e^x$

Slope of the tangent at $(0,4)$ is

$\left(\frac{dy}{dx}\right)_{(0,4)}=2(0)+2e^{(0)}=0+2(1)=2$

Equation of tangent at (x_1, y_1) is

$y-y_1=\left(\frac{dy}{dx}\right)_{(x_1, y_1)}(x-x_1)$

Here, $(x_1, y_1) \equiv(0,4)$

$\therefore$ Equation of the tangent at $(0,4)$ is

$(y-4)=2(x-0)$

$\therefore y-4=2x$

$\therefore 2x-y+4=0$

25. Since, there are six patients

$n=6$

$p=P(\text{ success })=0.5$ and $q=1-p=0.5$

$X \sim B(n=6, p=0.5)$

The p.m.f. of X is given by

$P(X=x)=p(x)={ }^6C_x(0.5)^x(0.5)^{6-x}$,

$x=0,1,2,3,4,5,6$

i. P (none will recover)

$= P(X=0) = {}^6C_0(0.5)^0(0.5)^{6-0} \ = (1)(1)(0.5)^6 = 0.015625$

ii. P(half of them will recover)

$= P(X=3) = {}^6C_3(0.5)^3(0.5)^{6-3} \ = \frac{6!}{3!.3!}(0.5)^3(0.5)^3$

$= \frac{6\times5\times4\times3!}{3\times2\times1\times3!}(0.125)(0.125)$

$= 20 \times 0.015625$

$= 0.3125$

26. i. The table gives a probability distribution and therefore

$\sum_{x=0}^{4} P(X=x) = 1$

$\therefore 0.1 + K + 2\,K + 2\,K + K = 1$

$\therefore 6\,K = 0.9$

$\therefore K = 0.15$

ii. $P(X<2) = P(X=0) + P(X=1)$

$= 0.1 + K$

$= 0.1 + 0.15 = 0.25$

iii. $P(X \geq 3) = P(X=3) + P(X=4)$

$= 2\,K + K = 3\,K$

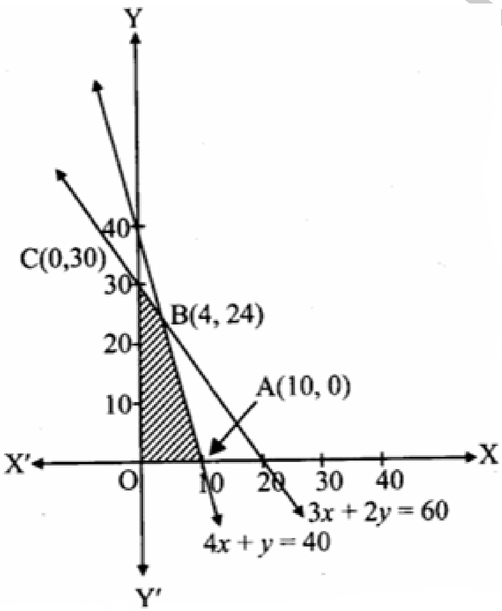
$= 3(0.15) = 0.45$

Section D

27. To draw the feasible region, construct table as follows:

Inequality	$4x + y \leq 40$	$3x + 2y \leq 60$
Corresponding equation (of line)	$4x + y = 40$	$3x + 2y = 60$
Intersection of line with X-axis	$(10, 0)$	$(20, 0)$
Intersection of line with Y-axis	$(0, 40)$	$(0, 30)$
Region	Origin side	Origin side

Shaded portion OABC is the feasible region, whose vertices are $O(0,0)$, $A(10,0)$, B and $C(0,30)$.



B is the point of intersection of the lines $4x + y = 40$ and $3x + 2y = 60$

Solving, we get

$B \equiv (4, 24)$

Here, the objective function is $Z = 150x + 250y$

Z at $O(0,0) = 150(0) + 250(0) = 0$

Z at $A(10,0) = 150(10) + 250(0) = 1500$

Z at $B(4,24) = 150(4) + 250(24) = 6600$

Z at $C(0,30) = 150(0) + 250(30) = 7500$

$\therefore Z$ has maximum value 7500 at $C(0,30)$.

$\therefore Z$ is maximum when $x=0$ and $y=30$.

28. Let the three numbers be x, y and z respectively. Now according to the first condition, we have $x+y+z=9$

According to the second condition, we have $3z+y=16$ i.e., $y+3z=16$

According to the third condition, we have

$x+z-2y=6$ i.e, $x-2y+z=6$

Matrix form of the above system of equations is,

$$\begin{bmatrix}1 & 1 & 1\\0 & 1 & 3\\1 & -2 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}9\\16\\6\end{bmatrix}$$

Applying $R_3 \rightarrow R_3-R_1$, we get

$$\begin{bmatrix}1 & 1 & 1\\0 & 1 & 3\\0 & -3 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}9\\16\\-3\end{bmatrix}$$

Applying $R_3 \rightarrow R_3+3R_2$, we get

$$\begin{bmatrix}1 & 1 & 1\\0 & 1 & 3\\0 & 0 & 9\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}9\\16\\45\end{bmatrix}$$

Hence, the original matrix is reduced to upper triangular matrix.

$\therefore$ We have by equality of matrices

$x+y+z=9$ ...(i)

$y+3z=16$ ...(ii)

$9Z=45$...(iii)

i.e., $z=5$

Substituting $z=5$ in equation (ii), we get

$y+3(5)=16$

$\therefore y+15=16$

$\therefore y=1$

Substituting $z=5, y=1$ in equation (i), we get

$x+1+5=9$

$\therefore x+6=9$

$\therefore x=3$

Hence, the three required numbers are $3,1,5$.

29. Let $\bar{a}, \bar{b}, \bar{c}$ be the position vectors of points A, B, C respectively of $\triangle ABC$ and $\bar{g}$ be the position vector of its centroid G.

$\therefore \bar{a}=5\hat{i}+\hat{j}+p\hat{k}, \bar{b}=\hat{i}+q\hat{j}+p\hat{k}, \bar{c}=\hat{i}-2\hat{j}+3\hat{k}$ and $\bar{g}=r\hat{i}-\frac{4}{3}\hat{j}+\frac{1}{3}\hat{k}$

$\therefore$ By using centroid formula,

$\bar{g}=\frac{\bar{a}+\bar{b}+\bar{c}}{3}$

$\therefore 3\bar{g}=\bar{a}+\bar{b}+\bar{c}$

$\therefore 3\left(\hat{r}\hat{i}-\frac{4}{3}\hat{j}+\frac{1}{3}\hat{k}\right)=(5\hat{i}+\hat{j}+p\hat{k})+(\hat{i}+q\hat{j}+p\hat{k})+(\hat{i}-2\hat{j}+3\hat{k})$

$\therefore 3r\hat{i}-4\hat{j}+\hat{k}=7\hat{i}+(q-1)\hat{j}+(2p+3)\hat{k}$

$\therefore$ By equality of vectors, we get

$3r=7, -4=q-1$ and $1=2p+3$

$\therefore r=\frac{7}{3}, q=-3$ and $p=-1$

30. The shortest distance between the lines

$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and

$\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ is given as

$$d=\frac{\left|\begin{matrix}x_2-x_1 & y_2-y_1 & z_2-z_1\\a_1 & b_1 & c_1\\a_2 & b_2 & c_2\end{matrix}\right|}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}$$

Here we have $x_1=1, y_1=2, z_1=3$

$x_2=2, y_2=4, z_2=5$

$a_1=2, b_1=3, c_1=4$
$a_2=3, b_2=4, c_2=5$
$\therefore$ Substituting we get

$$d=\frac{\begin{vmatrix} 2-1 & 4-2 & 5-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}}{\sqrt{[(3)(5)-(4)(4)]^2+[(4)(3)-(5)(2)]^2+[(2)(4)-(3)(3)]^2}}$$

$$\therefore d=\frac{\begin{vmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}}{\sqrt{(15-16)^2+(12-10)^2+(8-9)^2}}$$

$$=\left|\frac{1(15-16)-2(10-12)+2(8-9)}{\sqrt{(-1)^2+2^2+(-1)^2}}\right|$$

$$=\left|\frac{-1+4-2}{\sqrt{1+4+1}}\right|$$

$$=\left|\frac{1}{\sqrt{6}}\right|$$

$\therefore$ Shortest distance between given line is $d=\frac{1}{\sqrt{6}}$ units

31. Proof:

x and y are differentiable functions of t.
Let there be a small increment δt in the value of t.
Correspondingly, there should be a small increments $\delta x, \delta y$ in the values of x and y respectively.
As $\delta t \rightarrow 0, \delta x \rightarrow 0, \delta y \rightarrow 0$

Consider, $\frac{\delta y}{\delta x}=\frac{\frac{\delta y}{\delta t}}{\frac{\delta x}{\delta t}}$

Taking $\lim\limits_{\delta \rightarrow 0}$ on both sides, we get

$$\lim_{\delta \rightarrow 0} \frac{\delta y}{\delta x}=\frac{\lim\limits_{\delta t \rightarrow 0} \frac{\delta y}{\delta t}}{\lim\limits_{\delta t \rightarrow 0} \frac{\delta x}{\delta t}}$$

Since x and y are differentiable functions of t, $\lim\limits_{\delta \rightarrow 0} \frac{\delta y}{\delta t}=\frac{d y}{d t}$ exists and is finite.

$\lim\limits_{\delta \rightarrow 0} \frac{\delta x}{\delta t}=\frac{d x}{d t}$ exists and is finite.

$\therefore \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x}=\left(\frac{\frac{d y}{d t}}{\frac{d x}{d t}}\right)$ (as $\delta t \rightarrow 0, \delta x \rightarrow 0$)

$\therefore$ Limits on right hand side exists and are finite.
$\therefore$ Limits on the left hand side should also exists and be finite.

$\therefore \lim\limits_{\delta x \rightarrow 0} \frac{\delta y}{\delta x}=\frac{d y}{d x}$ exists and is finite.

$\therefore \frac{d y}{d x}=\left(\frac{\frac{d y}{d t}}{\frac{d x}{d t}}\right), \frac{d x}{d t} \neq 0$

$x=a \cos t, y=a \sin t$

$\therefore \frac{d x}{d t}=-a \sin t$ and $\frac{d y}{d t}=a \cos t$

$\therefore \frac{d y}{d x}=\left(\frac{\frac{d y}{d t}}{\frac{d x}{d t}}\right)=-\frac{a \cos t}{a \sin t}=-\cot t$

$x=a \cos ^2 t$ and $y=a \sin ^2 t$

$\therefore \frac{d x}{d t}=\frac{d}{d t}\left(a \cos ^2 t\right)=a[2 \cos t(-\sin t)]$

$\therefore \frac{d x}{d t}=-2 a \sin t \cos t$

and $\frac{d y}{d t}=\frac{d}{d t}\left(a \sin ^2 t\right)$

$\therefore \frac{d y}{d t}=a(2 \sin t \cos t)=2 a \sin t \cos t$

$\therefore \frac{d y}{d x}=\left(\frac{\frac{d y}{d t}}{\frac{d x}{d t}}\right)=\frac{2 a \sin t \cos t}{-2 a \sin cos t}=-1$

$x=\sin t$

$\therefore \frac{d x}{d t}=\frac{d}{d t}(\sin t)=\cos t$

$y=\cos t$

$\therefore \frac{dy}{dt}=\frac{d}{dt}(\cos t)=-\sin t$

$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{-\sin t}{\cos t}=-\tan t$

32. $f(x)=x^2-4x+10, x\in[0,4]$

As $f(x)$ is a polynomial function, it is continuous and differentiable everywhere on its domain. Thus,

1. $f(x)$ is continuous on $[0,4]$
2. $f(x)$ is differentiable on $(0,4)$

Further,

$f(0)=(0)^2-4(0)+10=10$

$f(4)=(4)^2-4(4)+10=16-16+10=10$

$\therefore f(0)=f(4)$

Thus, all the conditions of Rolle's theorem are satisfied.

The derivative of $f(x)$ should vanish for at least one point $c\in[0,4]$.

To obtain the value of c,

$f(x)=x^2-4x+10$

$\therefore f'(x)=2x-4$

$f'(c)=0$

$\therefore 2c-4=0$

$\therefore 2c=4$

$\therefore c=2$

$c=2$ lies between 0 and 4.

Thus, Rolle's theorem is verified.

33. Let $I=\int\frac{1}{\sin\theta+\sin 2\theta}d\theta$

$=\int\frac{1}{\sin\theta+2\sin\theta\cos\theta}d\theta$

$=\int\frac{d\theta}{\sin\theta(1+2\cos\theta)}=\int\frac{\sin\theta d\theta}{\sin^2\theta(1+2\cos\theta)}$

$=\int\frac{\sin\theta d\theta}{(1-\cos^2\theta)(1+2\cos\theta)}$

$=\int\frac{\sin\theta d\theta}{(1-\cos\theta)(1+\cos\theta)(1+2\cos\theta)}$

Put $\cos\theta=t$

$\therefore -\sin\theta d\theta=dt$

$\therefore \sin\theta d\theta=-dt$

$\therefore I=\int\frac{-dt}{(1-t)(1+t)(1+2t)}$

$=-\int\frac{dt}{(1-t)(1+t)(1+2t)}$

Let $\frac{1}{(1-t)(1+t)(1+2t)}=\frac{A}{1-t}+\frac{B}{1+t}+\frac{C}{1+2t}$

$\therefore 1=A(1+t)(1+2t)+B(1-t)(1+2t)+C(1-t)(1+t)$ ...(i)

Putting $t=1$ in (i), we get

$1=A(2)(3)+B(0)(3)+C(0)(2)$

$\therefore A=\frac{1}{6}$

Putting $t=-1$ in (i), we get

$1=A(0)(-1)+B(2)(-1)+C(2)(0)$

$\therefore B=-\frac{1}{2}$

Putting $t=-\frac{1}{2}$ in (i), we get

$1=A\left(\frac{1}{2}\right)(0)+B\left(\frac{3}{2}\right)(0)+C\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)$

$\therefore C=\frac{4}{3}$

$\therefore \frac{1}{(1-t)(1+t)(1+2t)}=\frac{\left(\frac{1}{6}\right)}{1-t}+\frac{\left(-\frac{1}{2}\right)}{1+t}+\frac{\left(\frac{4}{3}\right)}{1+2t}$

$\therefore I=-\int\left[\frac{\left(\frac{1}{6}\right)}{1-t}+\frac{\left(-\frac{1}{2}\right)}{1+t}+\frac{\left(\frac{4}{3}\right)}{1+2t}\right]dt$

$=-\frac{1}{6}\int\frac{1}{1-t}dt+\frac{1}{2}\int\frac{1}{1+t}dt-\frac{4}{3}\int\frac{1}{1+2t}dt$

$= -\frac{1}{6} \cdot \frac{\log|1-t|}{-1} + \frac{1}{2}\log|1+t| - \frac{4}{3} \cdot \frac{\log|1+2t|}{2} + c$

$\therefore I = \frac{1}{6}\log|1-\cos\theta| + \frac{1}{2}\log|1 + \cos\theta| - \frac{2}{3}\log|1+2\cos\theta| + c$

34. **Proof:**

Consider R.H.S.: $\int_a^b f(a+b-x)dx$

Let $I = \int_a^b f(a+b-x)dx$

Put $a+b-x=t$

$\therefore -dx = dt$

$\therefore dx = -dt$

When $x=a, t=a+b-a=b$

and when $x=b, t=a+b-b=a$

$\therefore I = \int_b^a f(t)(-dt)$

$= -\int_b^a f(t)dt$

$= \int_a^b f(t)dt$ ... $\left[\because \int_a^b f(x)dx = -\int_b^a f(x)dx\right]$

$= \int_a^b f(x)dx$ ... $\left[\because \int_a^b f(x)dx = \int_a^b f(t)dt\right]$

$=$ L.H.S.

$\therefore \int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Let $I = \int_a^b \frac{f(x)}{f(x)+f(a+b-x)}dx$ ...(i)

$= \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(a+b-(a+b-x))}dx$

$= \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(a+b-a-b+x)}dx$

$\therefore I = \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(x)}dx$ ...(ii)

Adding (i) and (ii), we get

$2I = \int_a^b \frac{f(x)+f(a+b-x)}{f(x)+f(a+b-x)}dx = \int_a^b 1dx = [x]_a^b$

$\therefore 2I = b-a$

$\therefore I = \frac{b-a}{2}$

$\therefore \int_a^b \frac{f(x)}{f(x)+f(a+b-x)}dx = \frac{b-a}{2}$