Solution

MATHEMATICS

Class 12 - Maths & Stats (Gen)

Section A

1. Select and write the correct answer for the following multiple-choice type of questions :

(i) **(b)** $\sim p \vee (q \wedge \sim r)$

Explanation: {

$\sim [p \wedge (q \rightarrow r)]$

$\equiv \sim p \vee \sim (q \rightarrow r) \ldots$ [De-Morgan's Law]

$\equiv \sim p \vee (q \wedge \sim r) \ldots$ [Negation of implication]

(ii) **(b)** 0

Explanation: {

$\sin^{-1}(1-x) - 2\sin^{-1} x = \frac{\pi}{2}$

Let $x = \sin\theta$

$\therefore \sin^{-1}(1-\sin\theta) - 2\sin^{-1}(\sin\theta) = \frac{\pi}{2}$

$\therefore \sin^{-1}(1-\sin\theta) - 2\theta = \frac{\pi}{2}$

$\therefore \sin^{-1}(1-\sin\theta) = \frac{\pi}{2} + 2\theta$

$\therefore 1 - \sin\theta = \sin\left(\frac{\pi}{2} + 2\theta\right)$

$\therefore 1 - \sin\theta = \cos 2\theta \ldots \left[\because \sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta\right]$

$\therefore 1 - \sin\theta = 1 - 2\sin^2\theta$

$\therefore 2\sin^2\theta - \sin\theta = 0$

$\therefore \sin\theta(2\sin\theta - 1) = 0$

$\therefore \sin\theta = 0$ or $\sin\theta = \frac{1}{2}$

$\therefore x = 0$ or $x = \frac{1}{2}$

For $x = \frac{1}{2}$,

L.H.S. $= \sin^{-1}(1-x) - 2\sin^{-1} x$

$= \sin^{-1}\left(1 - \frac{1}{2}\right) - 2\sin^{-1}\left(\frac{1}{2}\right)$

$= \sin^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\left(\frac{1}{2}\right)$

$= -\sin^{-1}\left(\frac{1}{2}\right)$

$= -\frac{\pi}{6} \neq$ R.H.S.

$\therefore x \neq \frac{1}{2}$

$\therefore x = 0$

(iii) **(b)** $xy - 3x - 2y + 6 = 0$

Explanation: {

Equations of the lines passing through the point $(2, 3)$ and parallel to the co-ordinate axes are $x = 2$ and $y = 3$ i.e., $x - 2 = 0$ and $y - 3 = 0$.

$\therefore$ The joint equation of these lines is

$(x-2)(y-3) = 0$

$\therefore xy - 3x - 2y + 6 = 0$

(iv) **(d)** T, F

Explanation: {

$p \rightarrow q = F \Rightarrow p \equiv T$ and $q \equiv F$

(v) **(d)** 1

Explanation: {

$\log(x+y) = 2xy$ ...(i)

Differentiating w. r. t. x, we get

$\frac{1}{x+y}(1+y') = 2(xy' + y)$

$\therefore 1 + y' = 2x(x+y)y' + 2y(x+y)$ ...(ii)

Substituting $x = 0$ in (i), we get

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$\log(0+y)=0$

$\therefore y(0)=1$

Substituting $x=0, y=1$ in (ii), we get

$1+y'(0)=0+2(1)(0+1)$

$\therefore y'(0)=2-1=1$

(vi) **(a)** $\frac{1}{4}$

Explanation: {

$\int \frac{dx}{4x^2-1}=\int \frac{dx}{(2x)^2-1^2}$

$=\frac{1}{2}\cdot\frac{1}{2(1)}\log\left|\frac{2x-1}{2x+1}\right|+c$

$=\frac{1}{4}\log\left|\frac{2x-1}{2x+1}\right|+c$

$\therefore A=\frac{1}{4}$

(vii) **(a)** x^2

Explanation: {

$\frac{dy}{dx}+\frac{2}{x}y=x\log x$

$\therefore$ I.F. $=e^{\int\frac{2}{x}dx}$

$=e^{2\log x}$

$=e^{\log x^2}$

$=x^2$

(viii) **(b)** 2.4

Explanation: {

$E(X)=np$

$\therefore 6=n(0.6)$

$\therefore n=10$

$q=1-p=1-0.6=0.4$

$\text{Var}(X)=npq$

$=(10)(0.6)(0.4)$

$=2.4$

2. Answer the following questions :

(i) $5x^2-9y^2=0$

$\therefore(\sqrt{5}x)^2-(3y)^2=0$

$\therefore(\sqrt{5}x+3y)(\sqrt{5}x-3y)=0$

$\therefore$ The separate equations of lines are $\sqrt{5}x+3y=0$ and $\sqrt{5}x-3y=0$.

(ii) Since the given vectors are collinear.

$\therefore\frac{2}{p}=\frac{-3}{6}=\frac{4}{-8}$

$\therefore\frac{2}{p}=\frac{-3}{6}$

$\therefore p=-4$

(iii) Let $I=\int\sec^n x\cdot\tan x\,dx$

$\therefore I=\int\sec^{n-1}\sec x\cdot\tan x\,dx$

Put $\sec x=t$

Differentiating w.r.t. x, we get

$\sec x\tan x\,dx=dt$

$\therefore I=\int t^{n-1}dt=\frac{t^{n-1+1}}{n-1+1}+c=\frac{t^n}{n}+c$

$\therefore I=\frac{\sec^n x}{n}+c$

(iv) $(y''')^2+3(y'')+3xy'+5y=0$

Here, the highest order derivative is (y''') i.e. $\frac{d^3y}{dx^3}$ with power 2.

By definition of order and degree,

order : 3 and degree : 2

Section B

3. $(p\wedge q)\vee(p\wedge r)$

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1	2	3	4	5	6
p	q	r	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	F	F
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F

Truth values in the last column are not identical.

$\therefore (p \wedge q) \vee (p \wedge r)$ is contingency.

4. Let $A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore |A| = \cos\theta(0-0) - (-\sin\theta)(0-0) + 0(0-0)$

$= 0 + 0 + 0$

$\therefore |A| = 0$

$\therefore A$ is a singular matrix.

$\therefore$ Inverse of the given matrix does not exist.

5. $\tan 2x = 0$

$\therefore \tan 2x = \tan 0$

Since, $\tan\theta = \tan\alpha$ implies $\theta = n\pi + \alpha, n \in Z$.

$\therefore 2x = n\pi + 0, n \in Z$

$\Rightarrow x = \frac{n\pi}{2}, n \in Z$

$\therefore$ The required general solution is $x = \frac{n\pi}{2}$, where $n \in Z$.

6. Let $\bar{r}$ be the position vector of the point R.

Given that R divides segment PQ internally in the ratio $2:1$.

$\therefore m : n = 2 : 1$

$\therefore$ By section formula, $\bar{r} = \frac{m\bar{q}+n\bar{p}}{m+n}$

$\therefore \bar{r} = \frac{2(\hat{i}+4\hat{j}-2\hat{k})+1(\hat{i}-2\hat{j}+\hat{k})}{2+1}$

$= \frac{2\hat{i}+8\hat{j}-4\hat{k}+\hat{i}-2\hat{j}+\hat{k}}{3}$

$= \frac{3\hat{i}+6\hat{j}-3\hat{k}}{3}$

$\therefore \bar{r} = \hat{i} + 2\hat{j} - \hat{k}$

7. Direction ratios of the given vector are $1, 2, -2$.

$\therefore$ Direction cosines of the given vector are

$\frac{1}{\sqrt{1^2+2^2+(-2)^2}}, \frac{2}{\sqrt{1^2+2^2+(-2)^2}}, \frac{-2}{\sqrt{1^2+2^2+(-2)^2}}$

i.e. $\frac{1}{3}, \frac{2}{3}, \frac{-2}{3}$

8. i. Let $p : 2$ is a rational number

$q : \sqrt{2}$ is an irrational number

$\therefore$ Symbolic form of the given statement is $p \wedge q$

The truth values of both p and q are T.

$\therefore$ Truth value of $p \wedge q$ is T.

ii. Let $p : 2 + 3 = 5$

$q : \sqrt{2} + \sqrt{3} = \sqrt{5}$

$\therefore$ Symbolic form of the given statement is $p \vee q$

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The truth value of p is T and the truth value of q is F.

$\therefore$ Truth value of $p \vee q$ is T.

9. $f(x) = (x-1)(x-2)(x-3), x \in [1,3]$

$= x^3 - 6x^2 + 11x - 6$

As $f(x)$ is a polynomial function, it is continuous and differentiable everywhere on its domain. Thus,

a. $f(x)$ is continuous on $[1,3]$

b. $f(x)$ is differentiable on $(1,3)$

Further, $f(1) = 0$ and $f(3) = 0$

$\therefore f(1) = f(3)$

Thus, all the conditions of Rolle's theorem are satisfied.

10. Let $I = \int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

$= \int_0^{\frac{\pi}{2}} \frac{1+\cos 2x}{2} \, dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 + \cos 2x) dx$

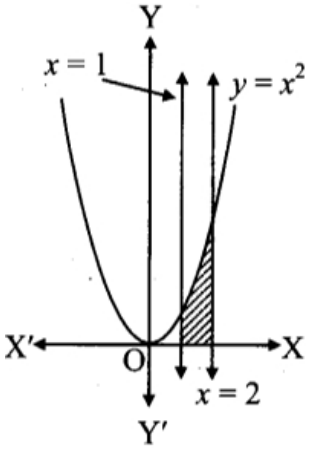
$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{2}\left[\left(\frac{\pi}{2} - 0\right) + \frac{1}{2}(\sin \pi - \sin 0)\right]$

$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{1}{2}(0 - 0)\right]$

$\therefore I = \frac{\pi}{4}$

11. Required area



$= \int_1^2 y dx$

$= \int_1^2 x^2 \, dx$

$= \frac{1}{3}\left[x^3\right]_1^2$

$= \frac{1}{3}\left(2^3 - 1^3\right) = \frac{1}{3}(8-1)$

$= \frac{7}{3}$ sq. units

12. $y\frac{dy}{dx} + x = 0$

$\therefore y\frac{dy}{dx} = -x$

$\therefore y \, dy = -x \, dx$

Integrating on both sides, we get

$\int y \, dy + \int x \, dx = 0$

$\therefore \frac{y^2}{2} + \frac{x^2}{2} = c_1$

$\therefore y^2 + x^2 = 2c_1$

$\therefore x^2 + y^2 = c$, where $c = 2c_1$

13. $E(X) = \sum x_i \cdot P(x_i)$

$= 0(0.45) + 1(0.35) + 2(0.15) + 3(0.03) + 4(0.02)$

$= 0 + 0.35 + 0.3 + 0.09 + 0.08$

$= 0.82$

$E\left(X^2\right) = \sum^2 x_i^2 \cdot P(x_i)$

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$= 0^2(0.45) + 1^2(0.35) + 2^2(0.15) + 3^2(0.03) + 4^2(0.02)$

$= 0 + 0.35 + 0.6 + 0.27 + 0.32$

$= 1.54$

$\therefore \text{Var}(X) = E\left(X^2\right) - [E(X)]^2$

$= 1.54 - (0.82)^2$

$= 1.54 - 0.6724$

$= 0.8676$

14. Let $I = \int_0^{\frac{\pi}{2}} \sqrt{1 - \cos 4x}\, dx$

$= \int_0^{\frac{\pi}{2}} \sqrt{2\sin^2 2x}\ dx$ ...$\left[\because 1 - \cos A = 2\sin^2 \frac{A}{2}\right]$

$= \sqrt{2}\int_0^{\frac{\pi}{2}} \sin 2x dx$

$= \sqrt{2}\left[\frac{-\cos 2x}{2}\right]_0^{\frac{\pi}{2}}$

$= -\frac{\sqrt{2}}{2}(\cos\pi - \cos 0)$

$= -\frac{\sqrt{2}}{2}(-1 - 1)$

$\therefore I = \sqrt{2}$

Section C

15. Let $\cos^{-1}\left(\frac{4}{5}\right) = x$

$\therefore \cos x = \frac{4}{5}$ and $0 < x < \frac{\pi}{2}$

$\therefore \sin x > 0$

Now, $\sin x = \sqrt{1 - \cos^2 x}$

$= \sqrt{1 - \left(\frac{4}{5}\right)^2}$

$= \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$

Let $\cos^{-1}\left(\frac{12}{13}\right) = y$

$\therefore \cos y = \frac{12}{13}$ and $0 < y < \frac{\pi}{2}$

$\therefore \sin y > 0$

Now, $\sin y = \sqrt{1 - \cos^2 y}$

$= \sqrt{1 - \left(\frac{12}{13}\right)^2} = \sqrt{1 - \frac{144}{169}} = \frac{5}{13}$

But $\cos(x + y) = \cos x \cos y - \sin x \sin y$

$= \frac{4}{5}\left(\frac{12}{13}\right) - \frac{3}{5}\left(\frac{5}{13}\right)$

$= \frac{48-15}{65} = \frac{33}{65}$

$\therefore x + y = \cos^{-1}\left(\frac{33}{65}\right)$

$\therefore \cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)$

16. Let $\sin^{-1} x = \theta$, where $x \in [-1, 1]$ and $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\therefore -\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\therefore \frac{\pi}{2} - \theta \in [0, \pi]$, which is the principal domain of the cosine function.

$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

$\therefore \cos\left(\frac{\pi}{2} - \theta\right) = x$

$\therefore \cos^{-1} x = \frac{\pi}{2} - \theta$

$\therefore \theta + \cos^{-1} x = \frac{\pi}{2}$

$\therefore \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

17. Auxiliary equation of $ax^2 + 2hxy + by^2 = 0$ is $bm^2 + 2hm + a = 0$.

Since one of the lines bisects an angle between the co-ordinate axes, the line will make an angle of 45° or 135° with the positive direction of X-axis.

$\therefore$ Slope of the line $= \tan 45^\circ$ or $\tan 135^\circ$

$\therefore m = \tan 45^\circ$ or $\tan(180^\circ - 45^\circ) = -\tan 45^\circ$

$\therefore m = 1$ or -1

$\therefore m = \pm 1$ are the roots of the auxiliary equation $bm^2 + 2hm + a = 0$.

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$\therefore b(\pm 1)^2 + 2h(\pm 1) + a = 0$

$\therefore b \pm 2h + a = 0$

$\therefore a + b = \mp 2h$

$\therefore (a+b)^2 = 4h^2$

18. $A \equiv (5, 5, \lambda), B \equiv (-1, 3, 2), C \equiv (-4, 2, -2)$

The direction ratios of line AB are

$-1-5, 3-5, 2-\lambda$, i.e., $-6, -2, 2-\lambda$

The direction ratios of line AC are

$-4-5, 2-5, -2-\lambda$, i.e., $-9, -3, -2-\lambda$

Since the points A, B, C are collinear,

the direction ratios of AB and AC are proportional.

$\therefore \frac{-6}{-9} = \frac{-2}{-3} = \frac{2-\lambda}{-2-\lambda}$

$\therefore \frac{2}{3} = \frac{2-\lambda}{-2-\lambda}$

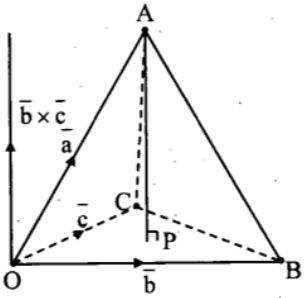
$\therefore -4 - 2\lambda = 6 - 3\lambda$

$\therefore 3\lambda - 2\lambda = 6 + 4$

$\therefore \lambda = 10$

19. Let $\overline{OA} = \bar{a}, \overline{OB} = \bar{b}$ and $\overline{OC} = \bar{c}$ be the coterminus edges of tetrahedron OABC.

Let AP be the height of the tetrahedron.



Volume of tetrahedron $= \frac{1}{3}$(area of $\triangle OCB$) $\times$ (Height AP)

But AP = Scalar projection of $\bar{a}$ on $(\bar{b} \times \bar{c})$

$= \frac{\bar{a} \cdot (\bar{b} \times \bar{c})}{|\bar{b} \times \bar{c}|} \ldots \left[\because \text{ Scalar projection of } \bar{p} \text{ on } \bar{q} \text{ is } \frac{\bar{p} \cdot \bar{q}}{|\bar{q}|}\right]$

Area of $\triangle OCB = \frac{1}{2}|\bar{b} \times \bar{c}|$

$\therefore$ Volume of tetrahedron $= \frac{1}{3} \times \frac{1}{2}|\bar{b} \times \bar{c}| \frac{\bar{a} \cdot (\bar{b} \times \bar{c})}{|\bar{b} \times \bar{c}|}$

$= \frac{1}{6}[\bar{a} \cdot (\bar{b} \times \bar{c})]$

$= \frac{1}{6}\left[\bar{a} \quad \bar{b} \quad \bar{c}\right]$

Volume of tetrahedron

$= \frac{1}{6}\left[\bar{a} \quad \bar{b} \quad \bar{c}\right]$

$= \frac{1}{6}\begin{vmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 2 & 1 & 4 \end{vmatrix}$

$= \frac{1}{6}[1(4-2) - 2(-4-4) + 3(-1-2)]$

$= \frac{1}{6}(2 + 16 - 9)$

$= \frac{1}{6}(9)$

$= \frac{3}{2}$ cubic units

20. The vector equation of the plane passing through the point $A(\bar{a})$ and parallel to the vectors $\bar{b}$ and $\bar{c}$ is $\bar{r} \cdot (\bar{b} \times \bar{c}) = \bar{a} \cdot (\bar{b} \times \bar{c})$.

Here, $\bar{a} = -\hat{i} + 2\hat{j} - 5\hat{k}, \bar{b} = 4\hat{i} - \hat{j} + 3\hat{k}, \ \bar{c} = \hat{i} + \hat{j} - \hat{k}$

$\therefore \bar{b} \times \bar{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 3 \\ 1 & 1 & -1 \end{vmatrix}$

$= (1-3)\hat{i} - (-4-3)\hat{j} + (4+1)\hat{k}$

$= -2\hat{i} + 7\hat{j} + 5\hat{k}$

$\therefore \bar{a} \cdot (\bar{b} \times \bar{c}) = (-\hat{i} + 2\hat{j} - 5\hat{k}) \cdot (-2\hat{i} + 7\hat{j} + 5\hat{k})$

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$= (-1)(-2) + 2(7) + (-5)(5)$

$= 2 + 14 - 25$

$= -9$

$\therefore$ The required equation of plane is $\bar{r} \cdot (-2\hat{i} + 7\hat{j} + 5\hat{k}) = -9$.

21. $\log_{10}\left(\frac{x^3-y^3}{x^3+y^3}\right) = 2$

$\therefore \frac{x^3-y^3}{x^3+y^3} = 10^2 = 100$

$\therefore x^3 - y^3 = 100\left(x^3 + y^3\right)$

$\therefore x^3 - y^3 = 100x^3 + 100y^3$

$\therefore -99x^3 - 101y^3 = 0$

Differentiating w.r.t. x, we get

$-99\left(3x^2\right) - 101\left(3y^2\frac{dy}{dx}\right) = 0$

$\therefore -99x^2 - 101y^2\frac{dy}{dx} = 0$

$\therefore -101y^2\frac{dy}{dx} = 99x^2$

$\therefore \frac{dy}{dx} = \frac{-99x^2}{101y^2}$

22. Let $I = \int \frac{x+1}{(x+2)(x+3)}dx$

Let $\frac{x+1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$

$\therefore x + 1 = A(x + 3) + B(x + 2)$ ...(i)

Putting $x = -3$ in (i), we get

$-3 + 1 = B(-3 + 2)$

$\therefore B = 2$

Putting $x = -2$ in (i), we get

$-2 + 1 = A(-2 + 3)$

$\therefore A = -1$

$\therefore \frac{x+1}{(x+2)(x+3)} = \frac{-1}{x+2} + \frac{2}{x+3}$

$\therefore I = \int \left(\frac{-1}{x+2} + \frac{2}{x+3}\right)dx = \int \left(\frac{2}{x+3} - \frac{1}{x+2}\right)dx$

$= 2\log|x + 3| - \log|x + 2| + c$

$= \log\left|(x + 3)^2\right| - \log|x + 2| + c$

$\therefore I = \log\left|\frac{(x+3)^2}{x+2}\right| + c$

23. $\frac{dy}{dx} = e^{2y}\cos x$

$\therefore \frac{dy}{e^{2y}} = \cos x\ dx$

$\therefore e^{-2y}\ dy = \cos x\ dx$

Integrating on both sides, we get

$\int e^{-2y}\ dy = \int \cos x\ dx$

$\therefore \frac{e^{-2y}}{-2} = \sin x + c$ ...(i)

When $x = \frac{\pi}{6}, y = 0$, we have

$\frac{e^0}{-2} = \sin\frac{\pi}{6} + c$

$\therefore -\frac{1}{2} = \frac{1}{2} + c$

$\therefore c = -1$

Substituting $c = -1$ in (i), we get

$\frac{e^{-2y}}{-2} = \sin x - 1$

$\therefore -e^{-2y} = 2\sin x - 2$

$\therefore e^{2y}(2\sin x - 2) + 1 = 0$, which is the required particular solution.

24. Let $f(x) = \tan^{-1} x$

$\therefore f'(x) = \frac{1}{1+x^2}$

$x = 1.002 = 1 + 0.002 = a + h$

Here, $a = 1$ and $h = 0.002$

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$f(a) = f(1) = \tan^{-1} 1 \ = \frac{\pi}{4} = \frac{3.1416}{4} = 0.7854$

$f'(a) = f'(1) = \frac{1}{1+1^2} = \frac{1}{2} = 0.5$

$f(a+h) \approx f(a) + hf'(a)$

$\therefore \tan^{-1}(1.002) \approx 0.7854 + (0.002)(0.5) \ \approx {}^{-1}.7854 + 0.001$

$\therefore \tan^{-1}(1.002) \approx 0.7864$

25. $X \sim B(n, p); E(X) = 6$

$\text{Var}(X) = 4.2$

Now $E(X) = np = 6$ ...(i)

and $\text{Var}(X) = npq = 4.2$ . ...(ii)

Dividing (ii) by (i), we get

$\frac{npq}{np} = \frac{4.2}{6}$

$\therefore q = \frac{42}{60}$

$\therefore q = 0.7$

$\therefore p = 1 - q$

$= 1 - 0.7$

$\therefore p = 0.3$

$np = 6$

$\therefore n = \frac{6}{p}$

$= \frac{6}{0.3}$

$\therefore n = 20$

26. Since the function $f(x)$ represents p.d.f. of r.v. X,

$\int_0^1 f(x)dx = 1$

$\therefore \int_0^1 kx(1-x)dx = 1$

$\therefore k\int_0^1 \left(x - x^2\right) dx = 1$

$\therefore k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1$

$\therefore k\left(\frac{1}{2} - \frac{1}{3}\right) = 1$

$\therefore \frac{k}{6} = 1$

$\therefore k = 6$

$P\left(X < \frac{1}{2}\right) = \int_0^{\frac{1}{2}} f(x)dx$

$= \int_0^{\frac{1}{2}} kx(1-x)dx$

$= 6\int_0^{\frac{1}{2}} \left(x - x^2\right) dx$

$= 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{\frac{1}{2}}$

$= 6\left[\frac{1}{2}\left(\frac{1}{4} - 0\right) - \frac{1}{3}\left(\frac{1}{8} - 0\right)\right]$

$= 6\left(\frac{1}{8} - \frac{1}{24}\right) = 6 \times \frac{1}{12}$

$\therefore P\left(X < \frac{1}{2}\right) = \frac{1}{2}$

Section D

27. To draw the feasible region, construct table as follows:

Inequality	$5x + 9y \leq 90$	$x + y \geq 4$	$y \leq 8$
Corresponding equation (of line)	$5x + 9y = 90$	$x + y = 4$	$y = 8$
Intersection of line with X-axis	$(18, 0)$	$(4, 0)$	-
Intersection of line with Y-axis	$(0, 10)$	$(0, 4)$	$(0, 8)$
Region	Origin side	Non-origin side	Origin side

Shaded portion $ABCDE$ is the feasible region, whose vertices are $A(4, 0), B(18, 0), C, D(0, 8)$ and $E(0, 4)$. C is the point of intersection of the lines $y = 8$ and $5x + 9y = 90$.

Putting $y = 8$ in $5x + 9y = 90$, we get

$5x + 72 = 90$

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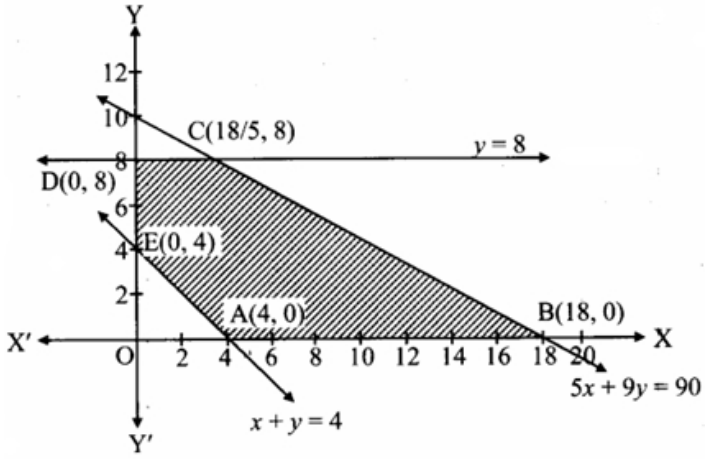
$\therefore x = \frac{18}{5}$

$\therefore C \equiv \left(\frac{18}{5}, 8\right)$



Here, the objective function is

$Z = 6x + 2y$,

Z at $A(4, 0) = 6(4) + 2(0) = 24$

Z at $B(18, 0) = 6(18) + 2(0) = 108$

Z at $C\left(\frac{18}{5}, 8\right) = 6\left(\frac{18}{5}\right) + 2(8)$

$= \frac{188}{5}$

$= 37.6$

Z at $D(0, 8) = 6(0) + 2(8) = 16$

Z at $E(0, 4) = 6(0) + 2(4) = 8$

$\therefore$ Z has minimum value 8 at $E(0, 4)$.

$\therefore Z$ is minimum, when $x = 0$ and $y = 4$.

28. Here we have

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

Now, $|A| = \begin{vmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{vmatrix}$

$= 1(16 - 9) - 3(4 - 3) + 3(3 - 4)$

$= 7 - 3(1) + 3(-1)$

$= 7 - 3 - 3$

$= 1$

$\therefore |A| \neq 0$

$\therefore A^{-1}$ exists

Now consider for column transformation

$A^{-1} A = I$

$$\therefore A^{-1} \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $C_2 \to C_2 - 3C_1$ and $C_3 \to C_3 - 3C_1$, we get

$$\therefore A^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $C_1 \to C_1 - C_2$

$$\therefore A^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & -3 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $C_1 \to C_1 - C_3$

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$\therefore A^{-1}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}7 & -3 & -3\\ -1 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}$

$\therefore A^{-1} = \begin{bmatrix}7 & -3 & -3\\ -1 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}$

29. Let $\bar{p}, \bar{q}$ and $\bar{r}$ be the position vectors of points P, Q and R respectively.

$\therefore \bar{p} = 4\hat{j}, \bar{q} = 3\hat{k}, \bar{r} = 4\hat{j} + 3\hat{k}$

If $H(\bar{h})$ is the incentre of $\triangle PQR$ then

$\bar{h} = \frac{|\overline{QR}|\bar{p} + |\overrightarrow{PR}|\bar{q} + |\overrightarrow{PQ}|\bar{r}}{|\overrightarrow{QR}| + |\overline{PR}| + |\overrightarrow{PQ}|}$ ...(i)

Now,

$\overline{QR} = \bar{r} - \bar{q} = (4\hat{j} + 3\hat{k}) - 3\hat{k} = 4\hat{j}$

$\therefore |\overline{QR}| = \sqrt{0^2 + 4^2 + 0^2} = \sqrt{4^2} = 4$

$\overline{PR} = \bar{r} - \bar{p} = (4\hat{j} + 3\hat{k}) - 4\hat{j} = 3\hat{k}$

$\therefore |\overrightarrow{PR}| = \sqrt{0^2 + 0^2 + 3^2} = \sqrt{3^2} = 3$

$\overline{PQ} = \bar{q} - \bar{p} = 3\hat{k} - 4\hat{j} = -4\hat{j} + 3\hat{k}$

$\therefore |\overline{PQ}| = \sqrt{0^2 + (-4)^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5$

$\therefore$ Substituting in equation (i), we get

$\bar{h} = \frac{4(4\hat{j}) + 3(3\hat{k}) + 5(4\hat{j} + 3\hat{k})}{4 + 3 + 5} = \frac{16\hat{j} + 9\hat{k} + 20\hat{j} + 15\hat{k}}{12}$

$\therefore \bar{h} = \frac{36\hat{j} + 24\hat{k}}{12}$

$\therefore \bar{h} = 3\hat{j} + 2\hat{k}$

$\therefore$ The required incentre is $(0, 3, 2)$.

30. The shortest distance between the lines

$\bar{r} = \overline{a_1} + \lambda\overline{b_1}$ and $\bar{r} = \overline{a_2} + \mu\overline{b_2}$ is

$d = \left|\frac{(\overline{a_2} - \overline{a_1}) \cdot \left(\overline{b_1} \times \overline{b_2}\right)}{|\overline{b_1} \times \overline{b_2}|}\right|$

Given lines are $\bar{r} = (4\hat{i} - \hat{j}) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k})$

and $\bar{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(\hat{i} + 4\hat{j} - 5\hat{k})$

Here, $\overline{a_1} = 4\hat{i} - \hat{j}, \overline{b_1} = \hat{i} + 2\hat{j} - 3\hat{k}$,

$\overline{a_2} = \hat{i} - \hat{j} + 2\hat{k}$ and $\overline{b_2} = \hat{i} + 4\hat{j} - 5\hat{k}$

Now, $\bar{a}_2 - \bar{a}_1 = (\hat{i} - \hat{j} + 2\hat{k}) - (4\hat{i} - \hat{j})$

$= -3\hat{i} + 2\hat{k}$

and $\overline{b_1} \times \overline{b_2} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & -3\\ 1 & 4 & -5\end{vmatrix}$

$= \hat{i}(-10 + 12) - \hat{j}(-5 + 3) + \hat{k}(4 - 2)$

$= 2\hat{i} + 2\hat{j} + 2\hat{k}$

$\therefore (\overline{a_2} - \overline{a_1}) \cdot \left(\overline{b_1} \times \overline{b_2}\right) = (-3\hat{i} + 2\hat{k}) \cdot (2\hat{i} + 2\hat{j} + 2\hat{k})$

$= -3 \times 2 + 0 \times 2 + 2 \times 2$

$= -6 + 0 + 4$

$= -2$

and $\left|\overline{b_1} \times \overline{b_2}\right| = \sqrt{2^2 + 2^2 + 2^2}$

$= \sqrt{4 + 4 + 4}$

$= \sqrt{12}$

$= 2\sqrt{3}$

$\therefore$ Shortest distance between the given lines $= \left|\frac{-2}{2\sqrt{3}}\right| = \frac{1}{\sqrt{3}}$ units

31. Proof:

x and y are differentiable functions of t.

Let there be a small increment δt in the value of t.

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Correspondingly, there should be a small increments $\delta x, \delta y$ in the values of x and y respectively.

As $\delta t \to 0, \delta x \to 0, \delta y \to 0$

Consider, $\frac{\delta y}{\delta x} = \frac{\frac{\delta y}{\delta t}}{\frac{\delta x}{\delta t}}$

Taking $\lim_{\delta \to 0}$ on both sides, we get

$\lim_{\delta \to 0} \frac{\delta y}{\delta x} = \frac{\lim_{\delta i \to 0} \frac{\delta y}{\delta t}}{\lim_{\delta t \to 0} \frac{\delta x}{\delta t}}$

Since x and y are differentiable functions of t, $\lim_{\delta \to 0} \frac{\delta y}{\delta t} = \frac{dy}{dt}$ exists and is finite.

$\lim_{\delta \to 0} \frac{\delta x}{\delta t} = \frac{dx}{dt}$ exists and is finite.

$\therefore \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)$ (as $\delta t \to 0, \delta x \to 0$)

$\therefore$ Limits on right hand side exists and are finite.

$\therefore$ Limits on the left hand side should also exists and be finite.

$\therefore \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$ exists and is finite.

$\therefore \frac{dy}{dx} = \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right), \frac{dx}{dt} \neq 0$

$x = a\cos t, y = a\sin t$

$\therefore \frac{dx}{dt} = -a\sin t$ and $\frac{dy}{dt} = a\cos t$

$\therefore \frac{dy}{dx} = \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = -\frac{a\cos t}{a\sin t} = -\cot t$

$x = a\cos^2 t$ and $y = a\sin^2 t$

$\therefore \frac{dx}{dt} = \frac{d}{dt}\left(a\cos^2 t\right) = a[2\cos t(-\sin t)]$

$\therefore \frac{dx}{dt} = -2a\sin t\cos t$

and $\frac{dy}{dt} = \frac{d}{dt}\left(a\sin^2 t\right)$

$\therefore \frac{dy}{dt} = a(2\sin t\cos t) = 2a\sin t\cos t$

$\therefore \frac{dy}{dx} = \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \frac{2a\sin t\cos t}{-2a\sin cost} = -1$

32. Let side of the square base be a cm and height be b cm.

$\therefore$ Surface area of the box $= a^2 + 4ab$

$\therefore 147 = a^2 + 4ab$

$\therefore b = \frac{147 - a^2}{4a}$ ... (i)

$\therefore$ Volume of the box $= a^2\, b$

$\therefore$ Volume of the box $= f(a) = a^2\left(\frac{147 - a^2}{4a}\right)$

$= \left(\frac{147a - a^3}{4}\right)$

$\therefore f'(a) = \left(\frac{147 - 3a^2}{4}\right)$

$\therefore f''(a) = \frac{-3a}{2}$

Consider, $f'(a) = 0$

$\therefore \frac{147 - 3a^2}{4} = 0$

$\therefore 147 - 3a^2 = 0$

$\therefore a^2 = 49$

$\therefore a = \pm 7$

But a cannot be negative.

$\therefore a = 7$

For $a = 7$,

$f''(a) = \frac{-3\times 7}{2} < 0$

$\therefore f(a)$, i.e., volume of the box is maximum when $a =$ side of the square base $= 7\ cm$ and

$b =$ height of the box $= \frac{147 - 49}{28} = 3.5\ cm$ . ...[From(i)]

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33. Let $I=\int \frac{1}{2+\cos x-\sin x}\,dx$

Put $\tan\left(\frac{x}{2}\right)=$ t

$\therefore x=2\tan^{-1}t$

$\therefore\ dx=\frac{2dt}{1+t^2}$ and $\sin x=\frac{2t}{1+t^2},\cos x=\frac{1-t^2}{1+t^2}$

$\therefore I=\int \frac{1}{2+\left(\frac{1-t^2}{1+t^2}\right)-\frac{2t}{1+t^2}}\times\frac{2dt}{1+t^2}$

$=\int \frac{2}{2+2t^2+1-t^2-2t}dt$

$=2\int \frac{1}{t^2-2t+3}dt$

$\left(\frac{1}{2}\text{ coefficient of }t\right)^2=\left(\frac{1}{2}\times-2\right)^2=(-1)^2=1$

$\therefore I=2\int \frac{1}{t^2-2t+1-1+3}dt$

$=2\int \frac{1}{(t-1)^2+2}dt$

$=2\int \frac{1}{(t-1)^2+(\sqrt{2})^2}dt$

$=2\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t-1}{\sqrt{2}}\right)+c$

$\therefore I=\sqrt{2}\tan^{-1}\left[\frac{\tan\left(\frac{x}{2}\right)-1}{\sqrt{2}}\right]+c$

34. Let $I=\int\limits_{-a}^{a}\sqrt{\frac{a-x}{a+x}}\,dx$

$=\int\limits_{-a}^{a}\sqrt{\frac{a-x}{a+x}\times\frac{a-x}{a-x}}\,dx$

$=\int\limits_{-a}^{a}\frac{a-x}{\sqrt{a^2-x^2}}\,dx$

$=\int\limits_{-a}^{a}\frac{a}{\sqrt{a^2-x^2}}\,dx-\int\limits_{-a}^{a}\frac{x}{\sqrt{a^2-x^2}}\,dx$

Now, $\frac{a}{\sqrt{a^2-x^2}}$ is an even function and

$\frac{x}{\sqrt{a^2-x^2}}$ is an odd function.

$\therefore I=2\int\limits_{0}^{a}\frac{a}{\sqrt{a^2-x^2}}\,dx-0$

$\ldots\left[\begin{array}{ll}\because \int_{-a}^{a}f(x)\mathrm{d}x & =2\int_{0}^{a}f(x)\mathrm{d}x,\ \text{if } f(x)\text{ is an even function}\\ & =0\quad,\ \text{if } f(x)\text{ is an odd function}\end{array}\right]$

$=2a\int\limits_{0}^{a}\frac{1}{\sqrt{a^2-x^2}}\,dx$

$=2a\left[\sin^{-1}\frac{x}{a}\right]_0^a$

$=2a\left(\sin^{-1}1-\sin^{-1}0\right)$

$=2a\left(\frac{\pi}{2}-0\right)$

$\therefore I=\pi a$

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